1. A \((0,1)\)-necklace of length \(n\) and weight \(i\) is a circular arrangement of \(i\) 1’s and \(n-i\) 0’s. For instance, the \((0,1)\)-necklaces of length 6 and weight 3 are (writing a circular arrangement linearly) 000111, 001011, 010011, and 010101. (Cyclic shifts of a linear word represent the same necklace, e.g., 000111 is the same as 110001.) Let \(N_n\) denote the set of all \((0,1)\)-necklaces of length \(n\). Define a partial order on \(N_n\) by letting \(u \preceq v\) if we can obtain \(v\) from \(u\) by changing some 0’s to 1’s. It’s easy to see (you may assume it) that \(N_n\) is graded of rank \(n\), with the rank of a necklace being its weight. Show that \(N_n\) is rank-symmetric, rank-unimodal, and Sperner. 

\textit{Hint:} Show that \(N_n \cong B_n/G\) for a suitable group \(G\).

2. How many necklaces (up to cyclic symmetry) have \(n\) read beads and \(n\) blue beads? Express your answer as a sum over all divisors \(d\) of \(n\).

3. Let \(\Gamma\) be the graph shown below.

\begin{center}
\begin{tikzpicture}[scale=0.7]
\node (1) at (0,0) {};
\node (2) at (1,1) {};
\node (3) at (2,2) {};
\node (4) at (3,1) {};
\node (5) at (2,-1) {};
\node (6) at (1,-2) {}; 
\draw (1) -- (2) -- (3) -- (4) -- (5) -- (6);
\end{tikzpicture}
\end{center}

An automorphism of \(\Gamma\) is a permutation \(\pi\) of the vertices of \(\Gamma\) that preserves adjacencies (i.e., there is an edge between two vertices \(x\) and \(y\) if and only if there is an edge between \(\pi(x)\) and \(\pi(y)\)). Let \(G\) be the automorphism group of \(\Gamma\), so \(G\) has order 8.

(a) What is the cycle index polynomial of \(G\), acting on the vertices of \(\Gamma\)?

(b) In how many ways can one color the vertices of \(\Gamma\) in \(n\) colors, up to symmetry of \(\Gamma\)?

4. For any finite group \(G\) of permutations of an \(\ell\)-element set \(X\), let \(f(n)\) be the number of inequivalent (under the action of \(G\)) colorings of \(X\) with \(n\) colors. Find \(\lim_{n \to \infty} f(n)/n^\ell\). Interpret your answer as saying that “most” colorings of \(X\) are asymmetric (have no symmetries).

5. Consider the group \(G\) of (orientation-preserving) symmetries of the cube.

(a) Show that \(|G| = 24\).

(b) Find the number of inequivalent colorings of the faces of the cube using \(n\) colors.

(c) Find the number of inequivalent colorings of the vertices of the cube using \(n\) colors.
6. Let $c(\lambda)$ denote the number of corner squares (or distinct parts) of the partition $\lambda$. For instance, $c(5, 5, 4, 2, 2, 1, 1) = 4$. Show that

$$
\sum_{\lambda \vdash n} c(\lambda) = p(0) + p(1) + \cdots + p(n - 1),
$$

where $p(i)$ denotes the number of partitions of $i$ (with $p(0) = 1$).