**Borsuk-Ulam Theorem via the Liftasaurus**

**Theorem 1** No continuous map

\[ f : S^2 \to S^1 \]

satisfies that \( f(-x) = -f(x) \) for every \( x \in S^2 \).

**Proof:** We will prove this by contradiction. To do so, let us suppose we have a continuous map

\[ f : S^2 \to S^1, \]

that preserves anti-podal points, or, in other words, satisfies that \( f(-x) = -f(x) \) for every \( x \in S^2 \). Recall that if we identify anti-podal points, then we form nice covers (the general case is exercise 1 from the exam). Let

\[ p : S^2 \to C^2 \]

be the covering map from \( S^2 \) to the cross surface \( C^2 \) that identifies anti-podal points; and let

\[ q : S^1 \to S^1 \]

be covering map from the circle to itself that identifies anti-podal points (given by the \( z^2 \) map from exercise 6 section 54 of Munkres). We have the following commutative diagram.

**Sub-lemma 1** Assuming there exist an \( f \) such that \( f(-x) = -f(x) \) for every \( x \in S^2 \), there exist a continuous map \( \tilde{f} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
S^2 & \xrightarrow{f} & S^1 \\
\downarrow{p} & & \downarrow{q} \\
C^2 & \xrightarrow{\tilde{f}} & S^1
\end{array}
\]

In other words, \( \tilde{f}p = qf \).

**Proof:** (Lemma [1]) We will be using lemma 22.2 of Munkres. Perhaps it is best to simply state who will play the role of all the object in Munkres’ lemma. The \( X \) from Munkres’ lemma will be our \( S^2 \). Munkres’ \( p \) is (fortunately) our quotient
map $p$, Munkres’ $Y$ is our $C^2$, Munkres’ $g$ is our continuous $qf$, and Munkres’ $Z$ is our $S^1$ in the lower right hand corner. Notice that
\[ g(x) = qf(x) = q(-f(x)) = qf(-x) = g(-x), \]
hence $g$ is constant on $p^{-1}(x) = \{x, -x\}$ as needed to utilize lemma 22.2 and assert that Munkres’ $f$, hence our needed $\tilde{f}$, exist, is continuous and satisfies our sought after $\tilde{f}p = g = qf$ condition.

\textbf{q.e.d}

Since $S^2$ is path connected, we may choose a path $\lambda$ that connects a pair of anti-podal points, $\{x, -x\} \subset S^2$. Following $\lambda$’s journey through the diagram in lemma[1] we have the following immediate consequence of our sublemma.

\textbf{Lemma 1}

\[ [\tilde{f}(p(\lambda))] = [q(f(\lambda))] \]

We will finish our proof off by contradicting this equality. Our contradiction will be that, as elements of $\pi_1(S^1, q(f(x)))$, that $[q(f(\lambda))]$ and $[\tilde{f}(p(\lambda))]$ are distinct. This will be an immediate consequence of the following pair of lemmas.

\textbf{Lemma 2}

\[ [\tilde{f}(p(\lambda))] = 0 \]

\textbf{Proof:} Notice $\lambda(0) = -\lambda(1)$, hence $p(\lambda)$ is loop at $p(x)$ in $C^2$. Recall that $\pi_1(C^2) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(S^1) = \mathbb{Z}$. Furthermore, notice that every homomorphism of $\mathbb{Z}/2\mathbb{Z}$ into the integers is the zero, since the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ must go to an element $m \in \mathbb{Z}$ with the property that $2m = 0$, hence $m = 0$. Now $\tilde{f}_*$ is such an homomorphism hence $\tilde{f}_* = 0$. In particular,
\[ 0 = \tilde{f}_*[p(\lambda)] = [\tilde{f}(p(\lambda))], \]
as needed. \textbf{q.e.d}

\textbf{Lemma 3}

\[ [q(f(\lambda))] \neq 0 \]

\textbf{Proof:} Notice, from the path lifting lemma that $f(\lambda)$ is the unique lift of $q(f(\lambda))$ starting at $f(x)$. Furthermore, notice since $f(\lambda)$ preserves anti-podal pairs that $f(\lambda)$ has distinct end points, namely
\[ f(\lambda)(0) = f(x) \neq -f(x) = f(\lambda)(1). \]
If \([q(f(\lambda))]\) were indeed equal to 0 in \(\pi_1(S^1, q(f(x)))\), then there would be a homotopy rel \(\{0, 1\}\) between \(q(f(\lambda))\) constant path \(q(f(x))\). By the homotopy lifting lemma, this homotopy would lift to a homotopy rel \(\{0, 1\}\) between \(f(\lambda)\) and the constant path \(f(x)\). Such a homotopy immediately contradicts the fact that \(f(\lambda)\) has distinct end points.

q.e.d

This same argument tells us some information about maps from \(S^1\) to itself. View the circle as \(R/\mathbb{Z}\) via the usual action of the integers, \(\phi(m)(x) = m + x\). Furthermore, from our key theorem this action gives us a canonical isomorphism between \(\pi_1(S^1, x)\) and \(\mathbb{Z}\). In particular, any continuous map

\[ f : S^1 \rightarrow S^1 \]

induces a mapping

\[ f_* : \pi_1(S^1, x) \rightarrow \pi_1(S^1, f(x)) \]

and hence a homomorphism

\[ f_* : \mathbb{Z} \rightarrow \mathbb{Z}. \]

Any homomorphism \(\psi\) from \(\mathbb{Z}\) to itself is given by \(\psi(m) = \text{deg}(f)m\) for some integer \(\text{deg}(f)\). Our next result is about this integer. Namely in the second problem of the final you will be asked to prove the following theorem.

**Theorem 2** If

\[ f : S^1 \rightarrow S^1 \]

satisfies that \(f(-x) = -f(x)\) for every \(x \in S^1\), then \(\text{deg}(f)\) is odd.