Math 74: Rotation Madness

Let $\mathbb{E}^n$ denote $\mathbb{R}^n$ with a choice of Euclidean inner product and an orientation. Let $SO(n)$ be the subgroup of the invertible linear transformations of $\mathbb{E}^n$ that preserve the Euclidean inner product and the orientation. Notice by choosing an orthonormal basis of $\mathbb{E}^n$ we can view $SO(n)$ as subset of the of the $n$ by $n$ matrices, $M_{n \times n}(\mathbb{R})$. In fact

**Theorem 1** Fixing an orthonormal pairs we have

$$SO(n) = \{ A \in M_{n \times n}(\mathbb{R}) \mid AA^t = I, \det(A) = 1 \}.$$ 

**Proof:** $A \in SO(n)$ will will sends a positively oriented orthonormal basis to another positively oriented orthonormal basis and in particular $A^t A = I$ is necessary since this is precisely this statement. Conversely any such matrix that satisfies $AA^t = I$ will preserve the inner product since $Av \cdot Aw = A^t Av \cdot w = v \cdot w$.

Recall a linear transformation $A$ preserves orientation if and only if $\det(A) > 0$. Since $\det(AA^t) = \det(A)^2 = \det(I) = 1$ we see that $\det(A) = \pm 1$, and the orientation preserving condition forces the $\det(A) = 1$.

**q.e.d.**

**Example 1:** $SO(2)$ For any $A \in SO(2)$, since $A$ preserves the notion of an oriented basis, we have that $B$ is determined by what $B$ does to single vector (by the right-hand rule). Viewing $A$ in an orthonormal basis $\{e_1, e_2\}$ we see that the fact that $A(e_1) = a_{11}e_1 + a_{21}e_2$ and since the norm is preserved $a_{11}^2 + a_{12}^2 = 1$ and hence $a_{11} = \cos(\theta)$ and $a_{12} = \sin(\theta)$ for some $\theta \in [0, 2\pi)$. By the right-hand rule, $A(e_2) = -\sin(\theta)e_1 + \cos(\theta)e_2$ and as a matrix

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$ 

Notice, in complete generality, we can view $SO(n)$ as subspace of the space of $n$ by $n$ matrices, $M_{n \times n}(\mathbb{R})$ which is topologically $\mathbb{R}^{n^2}$. Notice in its subspace topology, multiplication and inversion are continuous operations (in fact via the nice rational functions learned in linear algebra). Utilizing this topology, $SO(2)$ is in fact a very familiar topological space. Namely by the continuity of $\sin$ and $\cos$ the mapping of $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ to the above matrices is bijective and continuous. Hence, since $S^1$ is compact and and any subspace of $\mathbb{R}^{n^2}$ is Hausdorff, this mapping is a homeomorphism. Hence $SO(2)$ is homeomorphic to $S^1$ and in a particularly natural way.

**Example 2:** $SO(3)$ Notice, since we are in $\mathbb{E}^3$, if we fix a point $u \in S^2$, then the notion of a right-handed rotation about this axis makes sense via the right-hand rule. We can characterize the elements of $SO(3)$ as follows.
Theorem 2  Every $A \in SO(3)$ can described by fixing some point $u = (x, y, z) \in S^2$ and performing a right-handed rotation by some angle $\theta$ about $u$.

Proof: First observe we can explicitly write down the matrix that performs a right-handed rotation by an angle of $\theta$ about $u = (x, y, z) \in S^2$ via

$$R(u, \theta) = \begin{bmatrix}
1 + (1 - \cos(\theta))(x^2 - 1) & -z\sin(\theta) + (1 - \cos(\theta))xy & y\sin(\theta) + (1 - \cos(\theta))xz \\
z\sin(\theta) + (1 - \cos(\theta))xy & 1 + (1 - \cos(\theta))(y^2 - 1) & -x\sin(\theta) + (1 - \cos(\theta))yz \\
y\sin(\theta) + (1 - \cos(\theta))xz & x\sin(\theta) + (1 - \cos(\theta))yz & 1 + (1 - \cos(\theta))(z^2 - 1)
\end{bmatrix}.$$ 

By inspection $R(u, \theta) \in SO(3)$.

Given $A \in SO(3)$, we will first prove $A$ must fix a non-zero vector. Recall $\det(A^\text{tr}) = \det(A)$ hence

$$\det((I - A)A^\text{tr}) = \det(I - A) \det(A^\text{tr}) = \det(I - A).$$

By definition

$$(I - A)A^\text{tr} = (A^\text{tr} - I) = -(I - A^\text{tr}),$$

and by taking the determinant we have

$$\det(I - A)A^\text{tr}) = -\det(I - A^\text{tr}) = -\det(I - A),$$

and hence $\det(I - A) = 0$. This equivalent to $I - A$ having a nontrivial kernel, which implies there exist $v \neq 0$ such that $v - Av = 0$, as needed.

Take any $w$ in our fixed vector’s, $v$’s, orthogonal compliment. Note $Av = v$ hence $Aw$ is still orthogonal to to $v$. As such we may utilize a rotation, $R(v, \theta)$, about $v$ to send $Aw$ to $w$. Notice $R(v, \theta)A = B \in SO(3)$. For any $B \in SO(3)$ since $B$ preserves the notion of an oriented basis, we have that $B$ is determined by what $B$ does to any pair of orthogonal vectors (by the right-hand rule). By construction $R(v, \theta)A$ preserves two vector hence, from this observation, $R(v, \theta)A = I$. In other words $A = R(v, \theta)^{-1} = R(v, -\theta)$, as needed. q.e.d.

Once again in its subspace topology $SO(3)$ is a familiar space.

Theorem 3  $SO(3)$ is homeomorphic to $\mathbb{C}\mathbb{P}^3$.

Proof: First we will construct a map $\Psi$ from $B^3$ to $SO(3)$. Let

$$\Psi(x) = \begin{cases} 
R(x/|x|, \pi d(0, u)) & x \neq 0 \\
I & x = 0
\end{cases}.$$ 

By looking at the formula for $R(u, \theta)$ from the proof of theorem 2, we find that $\Psi$ is continuous. Now, since rotating by $\pi$ clockwise around an axis is the same
as rotating by $\pi$ counter-clockwise around this same axis, $\Psi$ is constant on $C^3$ equivalence classes. Hence (by theorem 2.22 of Munkres) we have a bijective continuous map from $C^3$ to $SO(3)$. The map is a homeomorphism since $C^3$ is compact and $SO(3)$ is Hausdorff. \textbf{q.e.d.}

Recall from the first problem of our first exam told us that $S^3 = \{(w, x, y, z) \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 1, \}$ naturally covers $C^3$ (notice the use of $(w, x, y, z)$ as coordinates of $\mathbb{R}^4$). In problem three of the final we will explore $S^3$ and end up with a much better understanding of this relationship between $S^3$ and $SO(3)$.

To do so it is convenient to observe that $S^3$ has some natural algebraic structures on it. First, we may identify $\mathbb{R}^4$ with $\mathbb{C}^2$ via the map sending $(w, x, y, z)$ to $(w + ix, y + iz)$. Hence $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. Recall if $z = x + iy$ then $\bar{z} = x - iy$, and $|z| = \sqrt{zz} = \sqrt{x^2 + y^2}$.

We may also identify $\mathbb{R}^4$ with $\mathbb{H}$, the quaternions, via the map sending $(w, x, y, z)$ to $w + xi + yj + zk$. Recall the quaternions are the vector space with basis \{1, i, j, k\}, formed into a ring (in fact a division ring), via the multiplication rule, $\ast$, which satisfies

$$i \ast i = -1, j \ast j = -1, k \ast k = -1$$

and

$$i \ast j = -j \ast i = k, j \ast k = -k \ast j = i, k \ast i = -i \ast k = j$$

on the basis elements and is extended by linearity to all of $\mathbb{H}$. As with the complex numbers, we have that if $q = w + xi + yj + zk$ then we let $\bar{q} = w - xi - yj - zk$, and $|q| = \sqrt{qq^*} = \sqrt{w^2 + x^2 + y^2 + z^2}$. Notice, by definition, $S^3$ is naturally equivalent to $U\mathbb{H} = \{q \mid |q| = 1\}$, the set of unit quaternions.