Key Theorem: Given a path connected, simply connect topological space $X$ and a properly discontinuous subgroup $G$ of $\text{Homeo}(X)$ (the homeomorphisms of $X$), then $\pi_1(X/G)$ is isomorphic to $G$.

1. For $n \geq 1$, let $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$, $B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, let $a : S^n \to S^n$ be the antipodal map defined by $a(x) = -x$, and let $C^n$ be the Cross Space viewed as the identification space formed by identifying the antipodal points of the boundary of $B^n$, namely the antipodal points of $S^{n-1}$.

(a) Prove that the the antipodal mapping generates a properly discontinuous subgroup, $G_n$, of $\text{Homeo}(S^n)$.

(b) Prove that $S^n / G_n$ is homeomorphic to $C^n$ (Please do this carefully using the ideas and results from section 22 of Munkres).

(c) Assuming $\pi_1(S^n) = \text{id}$ for $n \geq 2$, use the Key Theorem stated above to compute $\pi_1(C^n)$ for $n \geq 2$.

2. Let $A$ be the the loopy topologist’s sine curve. Namely the subspace of $\mathbb{R}^2$ determined by the points $\left(t, \sin \left( \frac{1}{t} \right) \right)$ for $t \in \left(0, \frac{1}{2\pi}\right]$ together with the three lines indicated in figure 1.

(a) Prove $A$ is simply connected.

(b) Construct a connected space $B$ such that $\text{Homeo}(B)$ contains a properly discontinuous subgroup, $G$, such that $G$ is isomorphic to the integers and such that $A$ is homeomorphic to $B/G$. (Hint: think about how the real line covers the circle).

(c) Explain why $B$’s existence does not contradict exercise 8 from section 54 of Munkres.
Figure 1: The loopy topologist’s sine curve

Figure 2: The Hawaiian earrings
Figure 3: An $\infty$-fold cover of the Hawaiian earrings

(d) Prove $\pi_1(B/G)$ is not isomorphic to $G$. Explain why this does not contradict the Key Theorem.

3. Let $Z$ denote the Hawaiian earrings from figure 2; namely, let $C_n$ be the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$ in $R^2$, and then let $Z = \bigcup_{n=1}^{\infty} C_n$ viewed as a subspace of $R^2$.

(a) Prove $Z$ is compact, path connected and locally path connected.

(b) For every open neighborhood $U$ of $(0, 0)$ in $Z$, prove that $\pi_1(U, (0, 0)) \neq id$.

(c) Demonstrate that there is a covering map $r : Y \rightarrow Z$ where $Y$ is the space described in figure 3.

(d) Construct a space $X$ and a covering map $q : X \rightarrow Y$ such that $p = r \cdot q : X \rightarrow Z$ is not a covering map. (Hint: You may restrict your search to $q$ which satisfy $|q^{-1}(y)| = 2$ for every $y \in Y$.)

(e) Explain why part ?? does not contradict exercise 4 from section 53 in Munkres.