Algebra Exam 1
Due Wednesday, February 20

Instructions You may use your class notes and homework for this exam. Don’t discuss the exam with anyone but me. If you find an error on the exam, let me know as soon as possible. I’ll e-mail everyone any corrections. Exams are due at the beginning of class on Wednesday, February 20.

1 Let $F \subseteq E$ be a Galois field extension. Assume that $G = \text{Gal}(E/F)$ is a simple group (recall that a simple group is one that has no proper normal subgroups besides the trivial subgroup).

(a) Let $\alpha \in E$. If $\alpha \notin F$, show that $E$ is a splitting field for $m_{F,\alpha}$.

(b) Let $H \subseteq G$ be a proper subgroup. Show that $E$ is a splitting field over $F$ for some polynomial $f \in F[X]$ with $\text{deg}(f) = |G:H|$.

(c) Assume further that $G$ is nonabelian. Show that all the roots of unity in $E$ (i.e. all $\alpha \in E$ such that $\alpha^n = 1$ for some $n$) actually lie in $F$.

2 Let $F \subseteq E$ be a field extension. Suppose $K$ and $L$ are intermediate fields with compositum $J = \langle K, L \rangle$. Assume that $K$ and $L$ are both Galois over $F$.

(a) Show that $J$ is Galois over $F$.

(b) If $\text{Gal}(K/F)$ and $\text{Gal}(L/F)$ are both abelian, show that $\text{Gal}(J/F)$ is also abelian,

(Hint: For (b), recall that the set of all commutators in $G$ generates a subgroup $G' \subseteq G$ with the property that $G/H$ is an abelian group if and only if $G' \subseteq H$. Now determine the fixed field of the commutator subgroup of $\text{Gal}(J/F)$.)
Let $F \subseteq E$ be an algebraic extension of fields. For this problem only, define

- $\alpha \in E$ is **Galois** over $F$ if $F[\alpha] \supseteq F$ is a Galois field extension;

- $\alpha \in E$ is **Abelian** over $F$ if $\alpha$ is Galois over $F$ and $\text{Gal}(F[\alpha]/F)$ is an abelian group.

Answer the questions True or False. If your answer is True, prove it. If your answer is False, give a counterexample.

(a) The set $K = \{\alpha \in E \mid \alpha \text{ is Galois}\}$ is a field.

(b) The set $K = \{\alpha \in E \mid \alpha \text{ is Abelian}\}$ is a field.

**Hint:** For (a), do you know of any specific elements $\alpha$ in some field that are *not* Galois? Can you think of a Galois extension without a primitive element?