LECTURE OUTLINE

Tangent Planes

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Math 8

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Goals

Tangent Planes
Linear Approximation
Example: \( f(x, y) = x^2 - y^2 \) at \((0, 0, 0)\).
Example: \( f(x, y) = x^2 - y^2 \) at \((0, 0, 0)\). Zoom in towards \((0, 0, 0)\).
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Tangent Plane

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Example: \( f(x, y) = x^2 - y^2 \) near \((0, 0, 0)\).
Example: $f(x, y) = x^2 - y^2$ near $(0, 0, 0)$. 
Tangent Plane

Provided a tangent plane exist, to find it we need two vectors. Argue that \(< 1, 0, \frac{\partial f}{\partial x}(a, b) >\) and \(< 0, 1, \frac{\partial f}{\partial y}(a, b) >\) should be in the tangent plane at \((a, b, f(a, b))\). From this the plane’s normal is

\[ \vec{n} = \langle -\frac{\partial f}{\partial x}(a, b) - \frac{\partial f}{\partial y}(a, b), 1 \rangle. \]

Example: Find the tangent plane of \(f(x, y) = x^2 - y^2\) at \((2, 1, 3)\).
Linear Approximation

We can approximate our function with this plane, namely

\[ f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b). \]

Example: Approximate the value of \( f(x, y) = x^2 - y^2 \) at \((2.05, 1.03)\). How close is it the true value?
A Cruel and UNUSUAL example

The tangent plane may fail to exist if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are not continuous. In other words, be careful when a denominator takes on a zero, or when function can’t make up its mind about a certain value.

Ex. Let $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}$. Find $f_x$ and $f_y$. Are they continuos?
The Non-Tangent Plane

Let \( f(x, y) = \frac{xy}{\sqrt{x^2+y^2}} \). Zoom in towards the \((0, 0, 0)\)...
The Non-Tangent Plane

Let $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$. Zoom in towards the $(0, 0, 0)$, and,...
The Non-Tangent Plane

Let $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}$. Zoom in towards the $(0, 0, 0)$, and nothing happens!
The Non-Tangent Plane

Let \( f(x, y) = \frac{xy}{\sqrt{x^2+y^2}} \) and zoom in towards \((0, 0, 0)\) on the graph of \( \frac{\partial f}{\partial x} \)...
The Non-Tangent Plane

Let \( f(x, y) = \frac{xy}{\sqrt{x^2+y^2}} \) and zoom in towards \((0, 0, 0)\) on the graph of \( \frac{\partial f}{\partial x} \)...
The Non-Tangent Plane

Let \( f(x, y) = \frac{xy}{\sqrt{x^2+y^2}} \) and zoom in towards \((0,0,0)\) on the graph of \( \frac{\partial f}{\partial x} \) and EEEEEEKKKKK!!!!!
The Differential

We can approximate our function with this plane. As such, near \((a, b)\) we have \(\Delta z = f(x, y) - f(a, b)\) is approximately

\[
\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.
\]

It can be useful to think using the differential

\[
dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.
\]

Example: Find the differential of \(f(x, y) = x^2 - y^2\) at \((2, 1, 3)\).
Limits

A function of two variables is called continuous at \((a, b)\) if

\[
\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b).
\]

We say \(f\) is continuous in \(D\) if it is continuous at each point of \(D\).

Example: Show \(f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}\) is continuous at zero, but that \(f_x\) is not.