LECTURE OUTLINE
Directional Derivative

Professor Leibon

Math 8

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Goals

Directional Derivative
Level Surfaces
Review

Let $\nabla f = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ (we called this the gradient) and $\vec{x} = \langle x_1, \ldots, x_n \rangle$ with each $x_i$ a function of the variables $t_1 \ldots t_m$, then we have the chain rule

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i}.$$

Example: An elliptical balloon described by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has volume given by $\frac{4\pi}{3}abc$. Suppose this balloon is being blown up so that when $(a, b, c) = (2, 4, 5)$ we have that $\frac{d}{dt}(a, b, c) = (0.1, 0.2, 0.25)$. At what rate is the balloon’s volume increasing when $(a, b, c) = (2, 4, 5)$?
Given a direction \( \hat{u} = a\hat{i} + b\hat{j} \), the **directional derivative** of \( f(x, y) \) at \((x_0, y_0)\) in the direction \( \hat{u} \) is

\[
\frac{df(x_0 + at, y_0 + bt)}{dt} = \nabla f(x_0, y_0) \cdot < a, b >.
\]

**Example:** Suppose the gradient of \( f \) at \((2, 3)\) is \(< -1, 5 >\). At what rate does \( f \) change as one heads in the direction \( \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} \) starting at \((2, 3)\)?
The Directional Derivative Function

Given a direction \( \hat{u} = a\hat{i} + b\hat{j} \) the \textit{directional derivative function} associated to \( f(x, y) \) is

\[
D_u f(x, y) = \frac{df(x + at, y + bt)}{dt} = \nabla f(x, y) \cdot < a, b >.
\]

Example: Find the directional derivative function associated to \( f(x, y) = ye^{x^2} \) in the direction \( \frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j} \).
Given a direction \( \hat{u} \) the *directional derivative function* associated to \( f \) in the direction \( \hat{u} \) is

\[
D_{\hat{u}} f = \nabla f \cdot \hat{u}.
\]

**Example:** Find the directional derivative function associated to \( f(x, y, z) = x^2 + y^2 + z^2 \) in the direction \( \hat{k} \).
Using the dot product

Let $\theta$ be the angle between $\hat{u}$ and $\nabla f$. Notice, we have

$$D_{u}f = |\nabla f| \cos(\theta)$$

Consequences: $f$ increases the fastest in the direction of $\nabla f$, $f$ decreases fastest the direction of $-\nabla f$, and $f$ does not change as we head in a direction perpendicular to $\nabla f$.

Contour...