Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$z = e^{xy} \tan y$, $x = s + 2t$, $y = \frac{s}{t}$

Finding our tree diagram:

So the chain rule for $\frac{\partial z}{\partial s}$ gives

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = ye^{xy} \tan y, \quad \frac{\partial z}{\partial y} = xe^{xy} \tan y + e^{xy} \sec^2 y$$

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = \frac{1}{t}$$

Thus:

$$\frac{\partial z}{\partial s} = ye^{xy} \tan y \left( \frac{\partial x}{\partial s} \right) + xe^{xy} \tan y + e^{xy} \sec^2 y \frac{\partial y}{\partial s}$$

$$= \frac{3}{t} e^{\frac{s}{t} + 1} \tan \left( \frac{s}{t} \right) + \frac{1}{t} e^{\frac{s}{t} + 1} \left( \frac{s}{t} \tan \left( \frac{s}{t} \right) + \sec^2 \left( \frac{s}{t} \right) \right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial t} = 2, \quad \frac{\partial y}{\partial t} = \frac{3}{t} \frac{s}{t} = -st^{-2} = -\frac{s}{t^2}$$

Hence:

$$\frac{\partial z}{\partial t} = \left( ye^{xy} \tan y \right) \left( \frac{3}{t} \right) + \left( xe^{xy} \tan y + e^{xy} \sec^2 y \right) \left( -\frac{s}{t^2} \right)$$

$$= \frac{5}{t} e^{\frac{s}{t} + 1} \left( 2 \tan \left( \frac{s}{t} \right) - \frac{s}{t} \tan \left( \frac{s}{t} \right) - \frac{1}{t} \right) \sec^2 \left( \frac{s}{t} \right)$$
If \( z = f(x, y) \), where \( f \) is differentiable, \( x = g(t) \), \( y = h(t) \),
\( g(3) = 2, \ g'(3) = 5, \ h(3) = 7, \ h'(3) = -4 \), \( f_x(2, 7) = 6 \) and
\( f_y(2, 7) = -8 \). Find \( \frac{dz}{dt} \) when \( t = 3 \).

The chain rule gives:
\[
\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt}
\]

Now
\[
\begin{align*}
\frac{dz}{dx} &= \frac{df}{dx} = f_x(x, y) \\
\frac{dx}{dt} &= \frac{dg}{dt} = g'(t) \\
\frac{dz}{dy} &= \frac{df}{dy} = f_y(x, y) \\
\frac{dy}{dt} &= \frac{dh}{dt} = h'(t)
\end{align*}
\]

so
\[
\frac{dz}{dt} = f_x(x, y)g'(t) + f_y(x, y)h'(t)
\]

from the statement of the problem at \( t = 3 \)
\( g(3) = 2, \ h(3) = 7 \) and \( g'(3) = 5, \ h'(3) = -4 \)

which implies
\[
\frac{dz}{dt} = f_x(2, 7) \cdot 5 + f_y(2, 7) \cdot (-4)
\]

also from the statement of the problem
\[ f_x(2, 7) = 6, \ f_y(2, 7) = -8 \]

thus
\[
\frac{dz}{dt} = 6 \cdot 5 + (-4)(-8) = 30 + 32 = 62
\]
Let \( W(s, t) = F(u(s, t), v(s, t)) \), where \( F, u, v \) are differentiable, \( u(1, 0) = 2, u_s(1, 0) = -2, u_t(1, 0) = 6, v(1, 0) = 3, v_s(1, 0) = 5, v_t(1, 0) = 4, F_u(2, 3) = -1 \), and \( F_v(2, 3) = 10 \).

Find \( u_s(1, 0) \) and \( v_t(1, 0) \).

\[
\begin{align*}
\frac{\partial W}{\partial t} &= \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial t} \\
\frac{\partial W}{\partial s} &= \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial s} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial s} \\
\end{align*}
\]

At \( (1, 0) \),
\[
\begin{align*}
u(1, 0) &= 2, & u_s(1, 0) &= -2 \\
v(1, 0) &= 3, & v_s(1, 0) &= 5 \\
F_u(u(1, 0), v(1, 0)) &= F_u(2, 3) = -1 \\
F_v(u(1, 0), v(1, 0)) &= F_v(2, 3) = 10 \\
\end{align*}
\]

So \( \frac{\partial W}{\partial t} |_{(1, 0)} = F_u(2, 3) u_s(1, 0) + F_v(2, 3) v_s(1, 0) \)
\[
= (-1)(6) + (10)(4) = -6 + 40 = 34
\]

and \( \frac{\partial W}{\partial s} |_{(1, 0)} = F_u(2, 3) u_t(1, 0) + F_v(2, 3) v_t(1, 0) \)
\[
= (-1)(-2) + (10)(5) = 2 + 50 = 52
\]
Use a tree diagram to write out the chain rule for the given case. Assume all functions are differentiable.

\[ v = f(p, q, r) \quad p = \phi(x, y, z) \quad q = \psi(x, y, z) \quad r = \rho(x, y, z) \]

\[
\frac{\partial v}{\partial x} = \frac{\partial v}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial v}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} \\
\frac{\partial v}{\partial y} = \frac{\partial v}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial v}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} \\
\frac{\partial v}{\partial z} = \frac{\partial v}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial v}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial z}
\]
Use the chain rule to find the indicated partial derivatives.

\[
u = \sqrt{r^2 + s^2} = (r^2 + s^2)^{1/2} \quad r = y + x \cos t \quad s = x + y \sin t
\]

\[
\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t} \quad \text{when} \quad x=1, y=2, t=0
\]

Things we will need:

\[
\frac{\partial r}{\partial s} = \frac{1}{2} (r^2 + s^2)^{-1/2} (2r) = r (r^2 + s^2)^{-1/2}
\]

\[
\frac{\partial u}{\partial s} = \frac{1}{2} (r^2 + s^2)^{-1/2} (2s) = s (r^2 + s^2)^{-1/2}
\]

\[
\frac{\partial r}{\partial x} = \cos t, \quad \frac{\partial r}{\partial y} = 1, \quad \frac{\partial r}{\partial t} = -x \sin t
\]

\[
\frac{\partial s}{\partial x} = 1, \quad \frac{\partial s}{\partial y} = x \sin t, \quad \frac{\partial s}{\partial t} = y \cos t
\]

Now

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} = r (r^2 + s^2)^{-1/2} \cos t + s (r^2 + s^2)^{-1/2} (1)
\]

\[
= \frac{(y + x \cos t)(\cos t) + (x + y \sin t)}{(\sqrt{(y + x \cos t)^2 + (x + y \sin t)^2})^{1/2}}
\]

So

\[
\frac{\partial u}{\partial x} \bigg|_{(1,2,0)} = \frac{(2 + 1)(1) + (1 + 2)(0)}{(\sqrt{(2 + 1)^2 + (1 + 2)^2})^{1/2}} = \frac{4}{\sqrt{10}} = \frac{4}{\sqrt{10}} = \frac{2\sqrt{10}}{5}
\]

\[
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} = \frac{\partial r}{\partial x} (1) + \frac{\partial s}{\partial x} (x \sin t)
\]

\[
= \frac{(y + x \cos t) + (x + y \cos t) \sin t}{(\sqrt{(y + x \cos t)^2 + (x + y \sin t)^2})^{1/2}}
\]

So

\[
\frac{\partial u}{\partial y} \bigg|_{(1,2,0)} = \frac{(2 + 1) + (1 + 2)(0)}{(\sqrt{(2 + 1)^2 + (1 + 2)^2})^{1/2}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}
\]
\[ \frac{du}{dt} = \frac{\partial u}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial u}{\partial s} \cdot \frac{ds}{dt} \]

\[ = \frac{(y + x \cos \theta) (-x \sin \theta) + (x+y \sin \theta) (y \cos \theta)}{(y + x \cos \theta)^2 + (x+y \sin \theta)^2} \frac{1}{\sqrt{e}} \]

so \[ \frac{du}{dt} \bigg|_{(1,7,0)} = \frac{(2+1)(-1)(0) + (1+2)(1)(2)(1)}{(2+1)^2 + (1+2)^2} \frac{1}{\sqrt{e}} = \frac{3}{\sqrt{10}} = \frac{\sqrt{10}}{5} \]
6) Use the chain rule to find the indicated partial derivatives.

\[ Y = \omega \tan^{-1}(uv), \quad u = r + s, \quad v = s + t, \quad \omega = t + r \]

\[ \frac{\partial Y}{\partial r}, \quad \frac{\partial Y}{\partial s}, \quad \frac{\partial Y}{\partial t} \quad \text{when} \quad r = 1, \quad s = 0, \quad t = 1 \]

Some things we will need.

\[ Y_u = \tan^{-1}(uv), \quad Y_v = \frac{1}{1 + u^2} \quad Y_\omega = \frac{1}{1 + v^2} \]

\[ \frac{\partial Y}{\partial r} = 1, \quad \frac{\partial Y}{\partial s} = 1, \quad \frac{\partial Y}{\partial t} = 0 \]

\[ \frac{\partial Y}{\partial r} = 0, \quad \frac{\partial Y}{\partial s} = 1, \quad \frac{\partial Y}{\partial t} = 1 \]

\[ \frac{\partial \omega}{\partial r} = 1, \quad \frac{\partial \omega}{\partial s} = 0, \quad \frac{\partial \omega}{\partial t} = 1 \]

\[ u(1, 0, 1) = 1, \quad v(1, 0, 1) = 1, \quad \omega(1, 0, 1) = 1 + 1 = 2 \]

\[ Y_u(1, 1, 2) = \frac{\pi}{4}, \quad Y_v(1, 1, 2) = \frac{\pi + 1}{4}, \quad Y_\omega(1, 1, 2) = 1 \]

So \[ \frac{\partial Y}{\partial r} = \frac{\partial Y}{\partial \omega} \cdot \frac{\partial \omega}{\partial r} + \frac{\partial Y}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial Y}{\partial v} \cdot \frac{\partial v}{\partial r} \]

Thus \[ \frac{\partial Y}{\partial r}(1, 0, 1) = \frac{\pi}{4} (1 + (1)(0) + (1)(0)) = \frac{\pi}{4} + 1 = \frac{\pi + 4}{4} \]

\[ \frac{\partial Y}{\partial s} = \frac{\partial Y}{\partial \omega} \cdot \frac{\partial \omega}{\partial s} + \frac{\partial Y}{\partial u} \cdot \frac{\partial u}{\partial s} + \frac{\partial Y}{\partial v} \cdot \frac{\partial v}{\partial s} \]

\[ = \frac{\pi}{4} (0) + (1)(1) + (1)(1) = 2 \]

\[ \frac{\partial Y}{\partial t} = \frac{\partial Y}{\partial \omega} \cdot \frac{\partial \omega}{\partial t} + \frac{\partial Y}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v} \cdot \frac{\partial v}{\partial t} \]

\[ = \frac{\pi}{4} (1) + (1)(0) + (1)(1) = \frac{\pi}{4} + 1 = \frac{\pi + 4}{4} \]
7) If \( z = f(x-y) \) show \( \frac{\partial^2 z}{\partial x} + \frac{\partial^2 z}{\partial y} = 0 \)

Let \( u = x-y \) the \( z = f(u) \)

hence \( \frac{\partial z}{\partial x} = f_u(x-y). \frac{\partial u}{\partial x} = -f_u(x-y) \)

\( \frac{\partial z}{\partial y} = f_u(x-y). \frac{\partial u}{\partial y} = f_u(x-y)(-1) = -f_u(x-y) \)

thus \( \frac{\partial^2 z}{\partial x} + \frac{\partial^2 z}{\partial y} = f_u(x-y) - f_u(x-y) = 0 \)
8) Show that any function of the form
\[ z = f(x + at) + g(x - at) \]
is a solution to the wave equation
\[ \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \]

**Proof:** Let \( u = x + at \) and \( v = x - at \)

then \( z = f(u) + g(v) \)

and \( \frac{\partial z}{\partial t} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial t} \)

Now \( \frac{\partial z}{\partial u} = f'(u) \) since \( g(v) \) does not depend on \( u \).

Similarly, \( \frac{\partial z}{\partial v} = g'(v) \)

Thus \( \frac{\partial z}{\partial t} = f'(u)(a) + g'(v)(-a) \)

Let \( h(u, v) = \frac{\partial z}{\partial t} \)

then \( \frac{\partial^2 z}{\partial t^2} = \frac{\partial h}{\partial t} = \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial t} \)

as above \( \frac{\partial h}{\partial u} = af''(u) \) since \( -a \frac{\partial g}{\partial v} = ag''(v) \)

Thus \( \frac{\partial^2 z}{\partial t^2} = af''(u)(a) - a \cdot ag''(v)(-a) = a^2 f''(u) + a^2 g''(v) = a^2 (f''(u) + g''(v)) \)

Similarly to the above,

\[ \frac{\partial^2 z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} \]

this time \( \frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial x} \)

hence \( \frac{\partial^2 z}{\partial x} = f'(u) + g'(v) \)

Similarly, let \( \ell(u, v) = \frac{\partial z}{\partial x} \)

then \( \frac{\partial^2 z}{\partial x^2} = \frac{\partial \ell}{\partial x} = \frac{\partial \ell}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \ell}{\partial v} \cdot \frac{\partial v}{\partial x} \)

\( \frac{\partial \ell}{\partial u} = f''(u), \quad \frac{\partial \ell}{\partial v} = g''(v) \)

hence \( \frac{\partial^2 z}{\partial x^2} = f''(u) + g''(v) \)

Thus \( \frac{a^2}{\partial^2 z/\partial x^2} = \frac{\partial^2 z/\partial t^2}{\partial x^2} \)
9) Find the directional derivative of $f$ at the given point in the direction given by the angle $\theta$

$$f(x, y) = \sqrt{5x - 4y} = (5x - 4y)^{1/2}, \quad (4, 1), \quad \theta = -\pi/6$$

A unit vector in the direction given by the angle $\theta$ is just $\langle \cos \theta, \sin \theta \rangle$.

$\theta = -\pi/6$, hence our vector is just

$$\langle \cos \frac{-\pi}{6}, \sin \frac{-\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle$$

Now from Theorem 3 in this section,

$$D_{\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle} f(x, y) = f_x(x, y) \left(\frac{\sqrt{3}}{2}\right) + f_y(x, y) \left(-\frac{1}{2}\right)$$

$$f_x(x, y) = \frac{1}{2} (5x - 4y)^{-1/2} \cdot 5$$

$$f_y(x, y) = \frac{1}{2} (5x - 4y)^{-1/2} \cdot (-4)$$

Hence

$$D_{\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle} f(4, 1) = \left(\frac{1}{2} \cdot \frac{1}{4} \cdot 5\right) \left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right) \left(\frac{1}{4}\right)(-4)(-\frac{1}{2})$$

$$= \frac{5\sqrt{3}}{16} + \frac{1}{16}$$
10) Find the directional derivative of $f$ at the given point in the direction indicated by the angle $\Theta$.

\[ f(x,y) = x \sin(xy) \quad (2,0) \quad \Theta = \pi/3 \]

A unit vector in the direction of $\frac{\pi}{3}$ is $\langle \cos \frac{\pi}{3}, \sin \frac{\pi}{3} \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

By theorem 3,

\[ D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(2,0) = f_x(2,0) \cdot \frac{1}{2} + f_y(2,0) \cdot \frac{\sqrt{3}}{2} \]

\[
\begin{align*}
  f_x(x,y) &= \sin(xy) + xy \cos(xy) \\
  f_y(x,y) &= x^2 \cos(xy) \\
  f_x(2,0) &= \sin(0) + 2(0) \cos(0) = 0 \\
  f_y(2,0) &= 4 \cos(0) = 4
\end{align*}
\]

Thus,

\[ D_{\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle} f(2,0) = 0 \cdot \frac{1}{2} + 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} \]
11. \( f(x,y) = y \ln x \quad P(1, -3) \quad \mathbf{u} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle \)

(a) Find the gradient of \( f \).
(b) Evaluate the gradient at the point \( P \).
(c) Find the rate of change of \( f \) at \( P \) in the direction of the vector \( \mathbf{u} \).

(a) The gradient of \( f = \langle f_x(x, y), f_y(x, y) \rangle \)

\[ f_x(x, y) = \frac{y}{x} \quad \text{and} \quad f_y(x, y) = \ln x \]

so the gradient of \( f \) is \( \left\langle \frac{y}{x}, \ln x \right\rangle \).

(b) \( \left\langle \frac{y}{x}, \ln x \right\rangle \mid (1, -3) = \left\langle -\frac{3}{1}, \ln (1) \right\rangle = \left\langle -3, 0 \right\rangle \)

(c) We must first find a unit vector in the direction of \( \mathbf{u} \).

To do this, we take \( \mathbf{u} \)

Now, \( ||\mathbf{u}|| = \sqrt{\left( -\frac{4}{5} \right)^2 + \left( \frac{3}{5} \right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = 1 \)

Thus, \( \mathbf{u} \) is a unit vector.

And to find the rate of change, we take

\[ \nabla f(1, -3) \cdot \mathbf{u} = \left\langle -3, 0 \right\rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = \frac{12}{5} \]
(a) Find the gradient of \( f \)

(b) Evaluate the gradient at the point \( P \)

(c) Find the rate of change of \( f \) at \( P \) in the direction of the vector \( u \)

\[
\nabla f(x, y, z) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right>
\]

\[
f_x = \frac{1}{2} (x+y+z)^{-1/2} (1)
\]

\[
f_y = \frac{1}{2} (x+y+z)^{-1/2} (2)
\]

\[
f_z = \frac{1}{2} (x+y+z)^{-1/2} (3)
\]

So \( \nabla f = \left< \frac{1}{2(x+y+z)} \right>, \left< \frac{2}{2(x+y+z)} \right>, \frac{y}{2(x+y+z)} \right> \)

\[
\nabla f(1, 3, 1) = \left< \frac{1}{2\sqrt{1+3+1}}, \frac{1}{2\sqrt{1}}, \frac{3}{4} \right> = \left< \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \right>
\]

\[
|u| = \sqrt{\frac{4}{9} + \frac{9}{49} + \frac{36}{99}} = \sqrt{\frac{4\times 9\times 99}{99}} = 1
\]

Hence \( u \) is a unit vector and the rate of change is given by

\[
\nabla f(1, 3, 1) \cdot u = \left< \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \right> \cdot \left< \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \right> = \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{28}
\]
Find the directional derivative of the function at the given point in the direction \( v \).

\[ g(s,t) = s^2 e^t \quad (2,0) \quad v = i + j \]

\( v \) is not a unit vector so we first have to find a unit vector in the direction of \( v \). This is just given by \( \frac{v}{|v|} \).

\[ |v| = \sqrt{1+1} = \sqrt{2} \quad \text{thus} \quad \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \text{ is such a vector} \]

\[
D_{\left< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right>} g(2,0) = g_s(2,0) \frac{1}{\sqrt{2}} + g_t(2,0) \frac{1}{\sqrt{2}}
\]

By theorem 3,

\[ g_s(2,0) = 2se^t \Rightarrow g_s(2,0) = 4 \]

\[ g_t(2,0) = s^2 e^t \Rightarrow g_t(2,0) = 4 \]

hence

\[
D_{\left< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right>} g(2,0) = \frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}
\]
Find the directional derivative of the function at the given point in the direction of \( \mathbf{v} \).

\[ f(x, y, z) = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}, \quad (1, 2, -2), \quad \mathbf{v} = \frac{1}{3}(-6, 6, 3) \]

First, we will find the unit vector in the direction of \( \mathbf{v} \):

\[ \|\mathbf{v}\| = \sqrt{36 + 36 + 9} = \sqrt{81} = 9 \]

Hence, \( \frac{\mathbf{v}}{\|\mathbf{v}\|} \) is a unit vector in the direction of \( \mathbf{v} \) and \( \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3}(-6, 6, 3) \)

We know from \([4]\) that \( D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u} \)

where \( \mathbf{u} \) is a unit vector.

\[ \nabla f(x, y, z) = \left< f_x, f_y, f_z \right> \]

\[ f_x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad f_y = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad f_z = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \]

So, \( \nabla f(1, 2, -2) = \left< \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right> \)

Hence, \( D_{\frac{\mathbf{v}}{\|\mathbf{v}\|}} f(1, 2, -2) = \left< \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right> \cdot \frac{1}{3}(-6, 6, 3) \]

\[ = \frac{1}{9}(-2 + 4 + 2) = \frac{4}{9} \]