The radius of a right circular cone is increasing at a rate of \(1.8 \text{ in/s}\) while its height is decreasing at a rate of \(2.5 \text{ in/s}\). At what rate is the volume of the cone changing when the radius is 120 in and the height is 140 in?

\[
V_{cone} = \frac{1}{3}\pi r^2 h
\]

\[
\frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt}
\]

\[
\frac{dV}{dt} = \frac{2\pi rh}{3} \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}
\]

\[
\frac{dV}{dt} = 2\pi (120)(140) \frac{1.8}{3} + \pi (120)^2 \frac{-2.5}{3}
\]

\[
\frac{dV}{dt} = 20160\pi - 12000\pi = 8160\pi \text{ in}^3/\text{s}
\]

2. Find the directional derivative \(f(x,y,z) = x^2 + y^2 + z^2 \) at \(P(2,1,3)\) in the direction of the origin.

\[
\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle
\]

\[
\nabla f(2,1,3) = \langle 4, 2, 6 \rangle
\]

\[
\vec{PO} = \langle -2, -1, -3 \rangle
\]

\[
|\vec{PO}| = \sqrt{(-2)^2 + (-1)^2 + (-3)^2} = \sqrt{14}
\]

\[
\frac{\nabla f}{|\nabla f|} = \frac{\langle 4, 2, 6 \rangle}{\sqrt{14}}
\]

3. Find the maximum rate of change of \(f\) at the given point and the direction in which it occurs.

\[
f(x,y) = \sqrt{x^2 + y^2}
\]

\[
\nabla f(x,y) = \langle \frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \rangle
\]

\[
\nabla f(2,4) = \langle \frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}} \rangle = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle
\]

\[
|\nabla f(2,4)| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}
\]

So the direction of the maximum rate of change is \(\langle -1, 1 \rangle\) and the rate of change is \(4\sqrt{2}\).
4. \[ f(x, y, z) = x^2y^3z^4 \quad (1, 1, 1) \]
\[ \nabla f(x, y, z) = \langle 2x^2y^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle \quad \nabla f(1, 1, 1) = \langle 2, 3, 4 \rangle \]
\[ |\nabla f(1, 1, 1)| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \]
So the maximum rate of change is \( \sqrt{29} \) in the direction of \( \langle 2, 3, 4 \rangle \)

5. \[ f(x, y, z) = \tan(x + 2y + 3z) \quad (-5, 1, 1) \]
\[ \nabla f(x, y, z) = \langle \sec^2(x + 2y + 3z), 2\sec^2(x + 2y + 3z), 3\sec^2(x + 2y + 3z) \rangle \]
\[ \nabla f(-5, 1, 1) = \langle \sec^2(0), 2\sec^2(0), 3\sec^2(0) \rangle = \langle 1, 2, 3 \rangle \]
\[ |\nabla f(-5, 1, 1)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \]
So the maximum rate of change is \( \sqrt{14} \) in the direction of \( \langle 1, 2, 3 \rangle \)

6. Show that a differentiable function \( f \) decreases most rapidly at \( x \) in the direction opposite to the gradient vector, i.e., in the direction of \( -\nabla f(x) \).

From the proof of Thm. 15, \( \Delta f = |f| \cos \theta \) when \( \theta = \pi \)
\( \cos \theta = -1 \), the minimum value. So the minimum value of \( \Delta f \) is \( -|f| \) occurring when \( \theta = \pi \), or when it is in the opposite direction of \( \nabla f \).

b. Find the direction in which \( f(x, y) = x^4 - x^2y^3 \) decreases fastest @ \( P(2, -3) \).
\[ \nabla f(x, y) = \langle 4x^3y - 2x^2y^3, x^4 - 3x^2y^2 \rangle \quad \nabla f(2, -3) = \langle 12, -9 \rangle \]
So the direction in which \( f(x, y) \) decreases the fastest @ \( P(2, -3) \)
is \( -\nabla f(2, -3) = \langle -12, 9 \rangle \).
7. Find the direction in which the directional derivative of \( f(x,y) = x^2 + \sin xy \) at the point \((1,0)\) has the value 1.

\[ f_x = 2x + y \cos xy, \quad f_y = x \cos xy \]

\[ f_x(1,0) = 2 \cdot 1 \cdot \cos 0 = 2, \quad f_y(1,0) = 1 \cdot \cos 0 = 1 \]

\[ D_{\mathbf{u}} f(1,0) = f_x(1,0) \cos \theta + f_y(1,0) \sin \theta = 2 \cos \theta + \sin \theta, \]

where \( \theta \) is the angle \( \mathbf{u} \) makes with the positive \( x \)-axis.

We want \( D_{\mathbf{u}} f(1,0) = 2 \cos \theta + \sin \theta = 1 \implies \sin \theta = 1 - 2 \cos \theta \)

\[ \sin^2 \theta = 1 - 4 \cos \theta + 4 \cos^2 \theta \implies 1 - \cos ^2 \theta = 1 - 4 \cos \theta + 4 \cos^2 \theta \]

\[ \implies 5 \cos^2 \theta - 4 \cos \theta = 0 \implies \cos \theta (5 \cos \theta - 4) = 0 \]

\[ \cos \theta = 0 \quad \text{or} \quad 5 \cos \theta - 4 = 0 \implies \cos \theta = \frac{4}{5} \]

\[ \implies \theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \cos^{-1} \left( \frac{4}{5} \right) \approx 53.13^{\circ} \]

8. Find the rate of change of \( T \) at \((1,2,2)\) in the direction toward the point \((2,1,3)\).

\[ T = \frac{K}{\sqrt{x^2 + y^2 + z^2}} \quad \text{and} \quad 120 = T(1,2,2) = \frac{K}{3} \implies K = 360. \]

\[ \mathbf{u} = \frac{\langle 2-1, 1-2, 3-2 \rangle}{\sqrt{(2-1)^2 + (1-2)^2 + (3-2)^2}} = \langle 1, -1, 1 \rangle \]

\[ \nabla T(1,2,2) = -360 \cdot \frac{1}{3} (x^2 + y^2 + z^2)^{-\frac{3}{2}} \langle 2x, 2y, 2z \rangle \]

\[ \nabla T \cdot \mathbf{u} = -40 \frac{1}{\sqrt{3}} \langle 1, -1, 1 \rangle = -40 \frac{1}{3 \sqrt{3}} (1 - 2 + 2) = -\frac{40}{3 \sqrt{3}} \]

8. Show that at any point in the ball, the direction of increase in temperature is given by a vector that points toward the origin.

\[ \langle x, y, z \rangle \] is the position vector of any point on the ball, so \( \langle -x, -y, -z \rangle \) always points toward the origin.

So \( \nabla T = 360 \langle -x, -y, -z \rangle \) also points toward the origin.
Sketch the gradient vector $\mathbf{V}f(4, 0)$ for the function $f$ whose level curves are shown. Explain how you choose the direction and length of this vector.

We know that $\mathbf{V}f(4, 0)$ is perpendicular to the level curve that includes $(4, 0)$. We estimate a portion of this level curve using the others. Then $\mathbf{V}f(4, 0)$ is perpendicular to this line at $(4, 0)$.

We need to estimate its length.

If we look at where $\mathbf{V}f(4, 0)$ would intersect the level curves $z = -2$ & $z = -3$ if we extended it, we would see that the two points of intersection are about $\frac{1}{2}$ apart. So the rate of change is approximately $\frac{-2 - (-3)}{\frac{1}{2}} = 2$. $\frac{1}{2}$

**5 (a)** Find the equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

(a) $x^2 + 2y^2 + 3z^2 = 21,(4, -1, 1)$

\[ F(x, y, z) = x^2 + 2y^2 + 3z^2 \] (So $x^2 + 2y^2 + 3z^2 = 21$ is just a level set of $F = 21$.)

\[
\begin{align*}
F_x &= 2x, & F_y &= 4y, & F_z &= 6z \\
F_x(4, -1, 1) &= 8, & F_y(4, -1, 1) &= -4, & F_z(4, -1, 1) &= 6
\end{align*}
\]

So our tangent plane at $(4, -1, 1)$ is

\[ 8(x - 4) + 4(y + 1) + 6(z - 1) = 0 \Rightarrow 8x - 32 - 4y - 4 + 6z - 6 = 0 \]

\[ \Rightarrow 8x - 4y + 6z = 42 \Rightarrow 4x - 2y + 3z = 21 \]

(b) From equation 20, we have the symmetric equations of the normal line:

\[
\begin{align*}
\frac{x - x_0}{F_x(x_0, y_0, z_0)} &= \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)} \\
\frac{x - 4}{8} &= \frac{y + 1}{-4} = \frac{z - 1}{6}
\end{align*}
\]

\[ \Rightarrow \frac{x - 4}{4} = \frac{y + 1}{2} = \frac{z - 1}{3} \]
11 \quad x - z = 4 \arctan(yz) \quad (-1, 1, 1)

(a) Let \( F(x, y, z) = x - z - 4 \arctan(yz) \) (Then \( x - z - 4 \arctan(yz) \) is a level surface of \( F \)).

\[ \nabla F(x, y, z) = \left< 1, -\frac{4}{1 + y^2 z^2}, -1 - \frac{4y}{1 + y^2 z^2} \right> \]

\[ \nabla F(1 + \pi, 1, 1) = \left< 1, -\frac{4}{1 + 1}, -1 - \frac{4}{1 + 1} \right> = \left< 1, -2, -3 \right> \]

So the equation for the tangent plane is

\[ 1(x - (1 + \pi)) + -2(y - 1) + -3(z - 1) = 0 \]

\[ \Rightarrow x - 1 - \pi - 2y + 2 - 3z + 3 = 0 \Rightarrow x - 2y - 3z = \pi - 4 \]

(b) So the normal line has symmetric equations:

\[ x - 1 - \pi = \frac{y - 1}{-2} = \frac{z - 1}{-3} \]