#1 Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - y + 3z = 1$.

$\nabla f(x_0, y_0, z_0) = \langle 2x_0, 4y_0, 6z_0 \rangle$ and $\langle 3, -1, 3 \rangle$ are both normal to the ellipsoid at $(x_0, y_0, z_0)$ where $(x_0, y_0, z_0)$ is a point where the tangent plane is parallel to $3x - y + 3z = 1$.

So we need $\langle 2x_0, 4y_0, 6z_0 \rangle = c \langle 3, -1, 3 \rangle$.

$\iff \langle x_0, 2y_0, 3z_0 \rangle = k \langle 3, -1, 3 \rangle$

So $x_0 = 3k$

$y_0 = \frac{1}{2}k$

$z_0 = k$

$\Rightarrow (3k)^2 + 2(\frac{1}{2}k)^2 + 3(k)^2 = k^2(9 + \frac{1}{2} + 3) = 1$

$\Rightarrow k = \pm \frac{1}{5}\sqrt{2}$

$\Rightarrow (x_0, y_0, z_0) = (\pm \frac{3\sqrt{2}}{5}, \mp \frac{1}{5}\sqrt{2}, \pm \frac{\sqrt{2}}{5})$

#2 Suppose $(1, 1)$ is a critical point of a function $f$ with continuous second derivatives. In each case, what can you say about $f$?

@ $f_{xx}(1, 1) = 4$, $f_{xy}(1, 1) = 1$, $f_{yy}(1, 1) = 2$

$D(1, 1) = f_{xx}(1, 1)f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = 4 \cdot 2 - 1^2 = 7 > 0$

$\Rightarrow$ $f_{xx}(1, 1) > 0$ so by the 2nd derivatives test $f$ has a local minimum at $(1, 1)$

B $f_{xx}(1, 1) = 4$, $f_{xy}(1, 1) = 3$, $f_{yy}(1, 1) = 2$

$D(1, 1) = f_{xx}(1, 1)f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = 4 \cdot 2 - 3^2 = -1 < 0$

$\Rightarrow f$ has a saddle point at $(1, 1)$ by the 2nd derivatives test.
Use the level curves in the figure to predict the location of the critical points of \( f \) and whether \( f \) has a saddle point or a local extrema at each of those points. Explain & check \( y \) 2nd derivatives test.

As we move away from \((-1,1) \& (-1,-1)\) in any direction, the values of \( f \) are increasing, so we expect local minima.

As we move away from \((1,0)\) in any direction, the values of \( f \) are decreasing, so we expect a local maximum. These are hyperbola-shaped level curves near \((-1,0), (1,1) \& (1,-1)\) and the values of \( f \) are decreasing as we move away in some directions and increasing in others, so we expect saddle points.

\[
f(x,y) = 3x^2 - x^3 - 2y^2 + y^4 \Rightarrow f_x(x,y) = 6x - 3x^2 \quad f_y = -4y + 4y^3\]

\[3 - 3x^2 < 0 \Rightarrow x = \pm 1 \quad -4y + 4y^3 = 0 \Rightarrow y(y^2 - 1) = 0 \Rightarrow y = 0 \text{ or } y = \pm 1\]

So the critical points are \((\pm 1, 0) \& (\pm 1, \pm 1)\).

\[
f_{xx} = -6x \quad f_{xy} = 0 \quad f_{yy} = 12y^2 - 4
\]

\[
D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2 - 4) - 0^2 = -72x y^2 + 24x
\]

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#4

Find the local max & min values & saddle points of the function.

\[ f(x,y) = x^4 + y^4 - 4xy + 2 \]

\[ f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x \]

\[ f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2 \]

\[ f_x = 0 \Rightarrow y = x^3 \quad \Rightarrow f_y = 4x^9 - 4x \]

\[ f_y = 0 \Rightarrow x(x^6 - 1) = 0 \Rightarrow x = 0 \text{ or } x = \pm 1 \quad \text{so critical points are } (0,0), (1,1), (-1,1) \]

\[ D(0,0) = 0 \cdot 0 - (-4)^2 = -16 < 0 \quad \Rightarrow (0,0) \text{ is a saddle point} \]

\[ D(1,1) = 12 \cdot 12 - (-4)^2 = 144 - 16 > 0 \quad f_{xx}(1,1) = 12 > 0 \]

\[ \Rightarrow (1,1) \text{ is a local min} \]

\[ D(-1,-1) = 12 \cdot 12 - (-4)^2 = 144 - 16 > 0 \quad f_{xx}(-1,-1) = 12 > 0 \]

\[ \Rightarrow (-1,-1) \text{ is a local min} \]

#5

\[ f(x,y) = xy(1-x-y) = xy - x^2y - xy^2 \]

\[ f_x = y - 2xy - y^2 \quad f_y = x - x^2 - 2xy \]

\[ f_{xx} = -2y \quad f_{xy} = 1 - 2x - 2y \quad f_{yy} = -2x \]

\[ f_x = 0 \Rightarrow y = 0 \text{ or } y = 1 - 2x \quad \Rightarrow f_y = x - x^2 \text{ or } 3y^2 - x = f_y \]

\[ f_y = 0 \Rightarrow x - x^2 = 0 \Rightarrow x = 0 \text{ or } x = 1 \]

\[ 3y^2 - x = 0 \Rightarrow x = 0 \text{ or } x = \frac{3}{2} \quad \text{so critical points are } (0,0), (1,0), (0,1), \left(0, \frac{1}{2}\right) \]

\[ D(0,0) = D(1,0) = D(0,1) = -1 < 0 \]

\[ \Rightarrow (0,0), (1,0), (0,1) \text{ are saddle points} \]

\[ D\left(\frac{3}{3}, \frac{1}{3}\right) = \frac{1}{3} \quad f_{xx}\left(\frac{3}{3}, \frac{1}{3}\right) = -\frac{2}{3} < 0 \]

\[ \Rightarrow f\left(\frac{3}{3}, \frac{1}{3}\right) = \frac{1}{2} \text{ is a local maximum} \]
#6 \[ f(x,y) = x^2 ye^{-x^2-y^2} \]

\[ f_x = \frac{x^2 y e^{-x^2-y^2} (-2x) + 2xy e^{-x^2-y^2}}{x^2 ye^{-x^2-y^2}} = 2xy \left( 1 - x^2 \right) e^{-x^2-y^2} \]

\[ f_y = \frac{x^2 ye^{-x^2-y^2} (-2y) + x^2 e^{-x^2-y^2}}{x^2 ye^{-x^2-y^2}} = x^2 \left( 1 - 2y^2 \right) e^{-x^2-y^2} \]

\[ f_{xx} = 2y \left( 2x^2 - 5x^2 + 1 \right) e^{-x^2-y^2} \]

\[ f_{xy} = 2x \left( 1 - x^2 \right) \left( 1 - 2y^2 \right) e^{-x^2-y^2} \]

\[ f_{yy} = 2x^2 ye^{-x^2-y^2} \]

\[ f_x = 0 \Rightarrow x = 0, y = 0 \text{ or } x = \pm 1 \]

\[ x = 0 \Rightarrow f_y = 0 \forall y \text{ so all } (0,y) \text{ are critical points.} \]

\[ y < 0 \Rightarrow f_y = 0 \Rightarrow x^2 e^{-x^2} = 0 \Rightarrow x = 0 \text{ so } (0,0) \text{ is a critical point.} \]

\[ x = \pm 1 \Rightarrow f_y = 0 \Rightarrow \left( 1 - 2y^2 \right) e^{-y^2} = 0 \Rightarrow y = \pm \frac{1}{\sqrt{2}} \]

So \( (1, \pm \frac{1}{\sqrt{2}}) \) and \( (-1, \pm \frac{1}{\sqrt{2}}) \) are critical points.

\[ D(0,y) = 0 \text{ so the second derivatives test tells us nothing, but when } y > 0, x^2 ye^{-x^2-y^2} > 0 \text{ and } = 0 \text{ only when } x = 0 \text{ so } f(0,y) = 0 \]

is a local min when \( y > 0 \) (Since everything around \( f(0,y) = 0 \) is \( > 0 \)).

\[ y < 0, x^2 ye^{-x^2-y^2} < 0 \text{ and } = 0 \text{ only when } x = 0 \text{ so } f(0,y) = 0 \text{ is a local max when } y < 0 \text{ (Since everything around } f(0,y) = 0 \text{ is } < 0) \]

And \((0,0)\) is a saddle point.

\[ D(\pm 1, \frac{1}{\sqrt{2}}) = 8e^{-3} > 0 \]

\[ f_{xx}(\pm 1, \frac{1}{\sqrt{2}}) = -2\sqrt{2} e^{-3/2} < 0 \]

So \( f(\pm 1, \frac{1}{\sqrt{2}}) = -\frac{1}{2} e^{-3/2} \) are local maxima.

\[ D(\pm 1, -\frac{1}{\sqrt{2}}) = 8e^{-3} > 0 \]

\[ f_{xx}(\pm 1, -\frac{1}{\sqrt{2}}) = 2\sqrt{2} e^{-3/2} > 0 \]

So \( f(\pm 1, -\frac{1}{\sqrt{2}}) = \frac{1}{2} e^{-3/2} \) are local minima.
#7
Find the absolute maximum & minimum values of \( f \) on the set \( D \).
\[ f(x, y) = 1 + 4x - 5y \quad D \text{ is the closed triangular region with vertices } (0, 0), (2, 0), \text{ and } (0, 3) \]

\( f \) is a polynomial so it is continuous on \( D \) and \( \text{max and min exist. } f_x = 4 \quad \text{ and } f_y = -5 \quad \text{ So there are no critical points inside } D \) \text{ and the absolute extremum must live on the boundary.} 

\( L_1 = (0, 0) \rightarrow (0, 3) \quad L_2 = (0, 0) \rightarrow (2, 0) \quad L_3 = (2, 0) \rightarrow (0, 3) \)

On \( L_1, \ x = 0 \Rightarrow f(0, y) = 1 - 5y \quad (0 \leq y \leq 3) \) which is a decreasing function so the maximum is \( f(0, 0) = 1 \) \text{ and the minimum is } \( f(0, 3) = -14 \)

On \( L_2, \ y = 0 \Rightarrow f(x, 0) = 1 + 4x \quad (0 \leq x \leq 2) \) which is an increasing function so the maximum is \( f(2, 0) = 9 \) and the minimum is \( f(0, 0) = 1 \)

On \( L_3, \ y = -\frac{3}{2}x + 3 \Rightarrow f(x, -\frac{3}{2}x + 3) = \frac{23}{2}x - 14 \quad (0 \leq x \leq 2) \)
which is an increasing function so the maximum is \( f(2, 0) = 9 \) and the minimum is \( f(0, 3) = -14 \)

So the absolute maximum is \( f(2, 0) = 9 \) and the absolute minimum is \( f(0, 3) = -14 \).

#8
Find three positive numbers whose sum is 100 & whose product is a maximum.
\[ x + y + z = 100 \quad \text{So we want to maximize } f(x, y, z) = xyz(100 - x - y) = 100xy - x^2y - xy^2 \]

\[ f_x = 100y - 2xy - y^2 \quad f_y = 100x - x^2 - 2xy \]

\[ f_{xx} = 2y \quad f_{xy} = 100 - 2x - 2y \quad f_{yy} = -2x \]

\( f_x = 0 \Rightarrow y = 0 \quad \text{ and } f_y = 0 \Rightarrow x = 0 \) or \( x = 100 \)

\( f_{xy} = 0 \Rightarrow 3x^2 - 100x = 0 \Rightarrow x = 0 \) or \( x = \frac{100}{3} \)

So the critical points are \( (0, 0), (100, 0), (10, 100), \) \& \( \left( \frac{100}{3}, \frac{100}{3} \right) \)
\[ D(0,0) = D(100,0) = D(0,100) = -10,000 < 0 \] so
\[ (0,0), (100,0) \text{ and } (0,100) \text{ are saddle points} \]
\[ D\left( \frac{100}{3}, \frac{100}{3} \right) = -\frac{2000}{3} < 0 \]
So \( \left( \frac{100}{3}, \frac{100}{3} \right) \) is a local maximum. So \[ x = y = z = \frac{100}{3} \]

9. Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm\(^2\).

Surface area \( = 2(xy + yz + xz) = 64 \) cm\(^2\) so \( xy + yz + xz = 32 \)

\[ \Rightarrow z = \frac{32 - xy}{x + y} \]
So we want to maximize \( f(x,y) = \frac{32 - xy}{x + y} xy \)

\[ f_x = \frac{32y - 2xy^2 - x^2y}{(x + y)^2} - y^2 \left( \frac{32 - 2xy - x^2}{(x + y)^2} \right) \]

\[ f_y = y^2 \left( \frac{32 - 2xy - x^2}{(x + y)^2} \right) \]

\[ f_x = 0 \Rightarrow y = \frac{32 - x^2}{2x} \] (since \( y \) cannot = 0)

\[ \text{and} \quad f_y = 0 \Rightarrow 32(4y^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0 \]

\[ \Rightarrow 3x^4 + 64x^2 - (32)^2 = 0 \]
\[ \Rightarrow x^2 = \frac{64}{6} \Rightarrow x = \frac{8}{\sqrt{6}} \Rightarrow y = \frac{64/3}{x} = \frac{8}{\sqrt{6}} \]

\[ z = \frac{32}{x} \]
Thus the box is the cube with edge length \( \frac{8}{\sqrt{6}} \) cm.
A cardboard box without a lid is to have a volume of 32,000 cm$^3$.
Find the dimensions that minimize the amount of cardboard used.
The surface area of the box is $xy + 2(xz + yz)$ if $xyz = 32,000$.

$z = \frac{32,000}{xy}$
So we wish to minimize

$$f(x, y) = xy + \frac{64,000(xy)}{xy} = xy + 64,000 \left(\frac{1}{x} + \frac{1}{y}\right)$$

$$f_x = y - 64,000x^2$$
$$f_y = x - 64,000y^2$$

$f_x = 0 \Rightarrow y = 64,000x^2$
Sub into $f_y = 0 \Rightarrow x^3 = 64,000 \Rightarrow x = 40 \Rightarrow y = 40$

$$D(x, y) = [(2)(64,000)]^2 x^3 y^{-3} - 1 > 0 @ (40, 40)$$

$f_x(40, 40) > 0$ so $f(40, 40)$ is a minimum and the box dimensions are $x = y = 40$ cm, $z = 20$ cm.