LECTURE OUTLINE
Space Curves

Professor Leibon

Math 8

Nov. 5, 2004
Goals

Vector functions
space curves
derivatives
integrals
Length
Position

We describe a particle’s position at time $t$ via a vector valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = < x(t), y(t), z(t) >,$$

with $x(t)$, $y(t)$, and $z(t)$ differentiable functions of $t$ for (sometimes we specify for $t$ is in a given interval $[a, b]$).

Example: Describe the space curve traced out by a particle following $\mathbf{u}(t) = < 2t + 3, 4t, -t + 7 >$.
Another Space Curve

Describe the space curve traced out by a particle following

\[ \vec{v}(t) = \langle \cos(t), \sin(t), t \rangle. \]

This curve is called a helix.
The Velocity Vector

\( \vec{r}'(t) \)'s instantaneous change at time \( t \), velocity, equals

\[
\lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d}{dt} \vec{r}'(t),
\]

2. Find the velocity at each time of particle traveling along a helix. Watch is the direction of this velocity, this is usually denoted \( \hat{T}(t) \) and called the unit tangent vector.
For $t \geq a$

$$\vec{r}'(t) = \int_a^t \frac{d}{dt} \vec{r}(t) \, dt + \vec{r}'(a)$$

where we integrate each component.

3. Describe the curves that share our helix’s velocity vector at each time.
Speed and Path Length

$\mathbf{r}'(t)$’s Speed is given by $|\frac{d}{dt} \mathbf{r}'(t)|$. while

$$L = \int_{a}^{b} |\frac{d}{dt} \mathbf{r}'(t)| \, dt$$

is the distance traveled while $t$ went from $a$ to $b$, in other words the curve’s length.

4. Find the length of our helix for $t$ in $[0, B]$. 
Differentiation Rules

1. \[ \frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \frac{d\mathbf{u}}{dt}(t) + \frac{d\mathbf{v}}{dt}(t) \]

2. \[ \frac{d}{dt}[c\mathbf{u}(t)] = c\frac{d\mathbf{u}}{dt}(t) \]

3. \[ \frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\frac{d\mathbf{u}}{dt}(t) + \frac{df}{dt}(t)\mathbf{u} \]

4. \[ \frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \frac{d\mathbf{u}}{dt}(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \frac{d\mathbf{v}}{dt}(t) \]

5. \[ \frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \frac{d\mathbf{u}}{dt}(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \frac{d\mathbf{v}}{dt}(t) \]

6. \[ \frac{d}{dt}[\mathbf{u}(f(t))] = \frac{d\mathbf{u}}{dt}(t)\frac{df}{dt}(t) \]
Here We Go!!!

Let \( \vec{v}(t) = \langle \cos(t), \sin(t), t \rangle \) and let \( \hat{T}(t) \) be its unit tangent vector. Show \( \hat{T} \cdot \frac{d}{dt} \hat{T} = 0 \) Explain why this is always true.

Find \( \frac{d}{dt} \hat{T}(t) \)'s direction vector for our helix, and call it \( \hat{N}(t) \) the curve's normal vector.

Let \( \hat{B}(t) = \hat{T}(t) \times \hat{N}(t) \), this is usually called the curve's binormal vector. Compute \( \hat{B} \) and \( \frac{d}{dt} \hat{B} \) for our helix. Why must \( \frac{d}{dt} \hat{B} \) be a multiple of \( \hat{N} \)?

Show that \( \frac{d}{dt} \hat{N}(t) = -a(t)\hat{T} + b(t)\hat{B} \), for our helix. Why is this always true?