LECTURE OUTLINE
Sequences and Limits

Professor Leibon
Math 8
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Goals

Improper Integrals
The Integral Comparison Test
Sequences
Improper Integral

If \( \int_a^t f(x) \, dx \) exists for every number \( t \geq a \), then

\[
\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx
\]

provided this limit exists (as a finite number). We say the integral is *convergent* if the limit exist and *divergent* otherwise.

Practice Example: \( \int_1^\infty \frac{1}{x^2} \, dx = \lim_{t \to \infty} (1 - \frac{1}{t}) = 1.\)
The Integral Comparison Test

Comparison Theorem: Suppose \( f(x) \) and \( g(x) \) are continuous functions with \( f(x) \geq g(x) \geq 0 \) for all \( x > a \).

(a) If \( \int_a^\infty f(x) \, dx \) is convergent, then \( \int_a^\infty g(x) \, dx \) is convergent.

(b) If \( \int_a^\infty g(x) \, dx \) is divergent, then \( \int_a^\infty f(x) \, dx \) is divergent.

Example: Decide whether \( \int_1^\infty \frac{(\cos(x))^2}{x^2} \, dx \) and \( \int_1^\infty \frac{3+e^{-2x}}{x} \, dx \) are divergent or convergent.
Improper Integral

If \( f(x) \) is continuous on \([a, b)\) and is discontinuous at \( b \), then

\[
\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx
\]

provided this limit exists (as a finite number), and we call the integral convergent.

Example: Decide whether \( \int_0^1 \frac{1}{\sqrt{1-x}} \, dx \) is divergent or convergent, and find its value if it is convergent.
A Sequence

A sequence is a list of numbers $a_1, a_2, a_3 \ldots, a_n \ldots$, often denoted as $\{a_1, a_2, a_3 \ldots\}$, $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$. 
A sequence \( \{a_n\} \) has limit \( L \) provided for every \( \varepsilon > 0 \) there exist an integer \( N \) such that for every \( n > N \)

\[ |a_n - L| < \varepsilon. \]
**A Convergent Sequence**

If \( \{a_n\} \) has a limit \( L \), we say \( \{a_n\} \) is *convergent* and we denote this as \( a_n \rightarrow L \) as \( n \rightarrow \infty \) or

\[
\lim_{n \to \infty} a_n = L.
\]

When \( \{a_n\} \) has no limit we call \( \{a_n\} \) *divergent*.

Example: Decide whether \( \{(−1)^n\} \) is convergent or divergent.
Sequences Given by a Formula

If \( \lim_{x \to \infty} f(x) = L \) and \( a_n = f(n) \), then

\[
\lim_{x \to \infty} a_n = L.
\]

Example: Find the limit of \( \left\{ \frac{n}{(n+1)^2} \right\} \).
Squeeze Theorem

The Squeeze Theorem: If $a_n \leq b_n \leq c_n$ for $n > N$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$, then $\lim_{n \to \infty} a_n = L$.

Corollary: If $\lim_{n \to \infty} |a_n| = 0$, then $\lim_{n \to \infty} a_n = 0$.

Example: Find the limit of $\{\frac{n!}{n^n}\}$ and $\{\frac{(-1)^n}{n}\}$.