Goals

Review Remainder Estimates
Manipulating Taylor Series
Radius of Convergence
Taylor’s Inequality

Given \( f(x) \) let the \( n \)th Taylor Expansion be
\[
T_N(x) = \sum_{n=0}^{N} \frac{f^n(a)}{n!}(x - a)^n,
\]
and let the \( N \)th Remainder be
\[
R_N(x) = f(x) - T_N(x).
\]

Theorem: Suppose \( |f^{n+1}(x)| \leq M \) for every \( x \) in \([a, x]\) if \( x > a \) (or \([x, a]\) if \( x < a \)), then
\[
|R_N(x)| \leq M \frac{|x - a|^{N+1}}{(N + 1)!}.
\]
We write $x = y \pm e$ to mean that $x$ is in the interval $[y - e, y + e]$. Taylor’s Inequality asserts

$$f(x) = T_N(x) \pm M \frac{|x - a|^{N+1}}{(N + 1)!}.$$

(12.12: 25.) Use Taylor’s Inequality to determine the number of terms of the Maclaurin series for $e^x$ that should be used to estimate $e^{0.1}$ to within 0.00001. (While we are at it, find a number $C$ so that $e^{0.1} = C \pm 0.00001$.)
A Different Sort of Example

(12. Review: 56) Use series to approximate

\[ \int_0^1 \sqrt{1 + x^4} \, dx \]

correct to two decimal places. (Due this with and without Taylor’s Inequality.)
Radius of Convergence

Recall, by the ratio test, $\sum_{n=0}^{\infty} c_n (x - a)^n$ will converge if

$$0 < \lim_{n \to \infty} \left| \frac{c_{n+1}(x-a)}{c_n} \right| < 1$$

and diverge if

$$1 < \lim_{n \to \infty} \left| \frac{c_{n+1}(x-a)}{c_n} \right| < \infty.$$ 

(12. Review: 44) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)(n!)^2} x^n.$$
(12. Review: 53) Find the Maclaurin series and radius of convergence of

\[ f(x) = (16 - x)^{-1/4}. \]