LECTURE OUTLINE

Dot product and cross product

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Math 8

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Dot Product

Rounding Review
The dot product
Scalar and Vector projection
Rounding 101

We will interpret "approximate the following NUMBER to $N$ digits" to mean that we are we know the NUMBER rounded to $N$ places. (This means we replace the NUMBER with the a number in the form $\frac{M}{10^N}$ such that $|\text{NUMBER} - \frac{M}{10^N}| \leq \frac{5}{10^{N+1}}$. Notice this leaves two possibilities when our number is in the form $\frac{K}{10^N} + \frac{5}{10^{N+1}}$. What do you usually do and why?).

Yesterday we found $\int_0^1 \sqrt{1 + x^4} dx = 1.090918803 \pm 0.002297794121$, or rather that $\int_0^1 \sqrt{1 + x^4} dx$ was somewhere in the interval $[1.08862101, 1.09321660]$ (by using the first 4 terms of the series and the remainder estimate for alternating series). Have we approximated $\int_0^1 \sqrt{1 + x^4} dx$ to 2 digits? How about to 3 digits?

Our first approximation (using 3 terms) was the observation that $\int_0^1 \sqrt{1 + x^4} dx$ was somewhere in the interval $[1.081303419, 1.090918803]$. Had we already approximated $\int_0^1 \sqrt{1 + x^4} dx$ to 2 digits?
Dot Product

Given two vectors $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$, then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_3 + a_3 b_3$$

Ex.: Let $\vec{a} = < -1, 2, 5 >$ and $\vec{b} = < 2, 2, 7 >$, and find $\vec{a} \cdot \vec{b}$. 
basic properties

\[ \vec{a} \cdot \vec{a} = |a|^2 \]

\[ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \]

\[ \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \]

\[ (c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b}) \]

Ex.: Let \( \vec{a} = \langle -1, 2, 5 \rangle \) and \( \vec{b} = \langle 2, 2, 7 \rangle \), and find \( (\vec{a} + 3\vec{b}) \cdot (2\vec{a} + \vec{b}) \).
The Big Fact

Letting $\theta$ be the angle between $\vec{a}$ and $\vec{b}$, we have

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Ex.: Let $\vec{a} = <-1, 2, 5>$ and $\vec{b} = <2, 2, 7>$, and find the angle between $\vec{a}$ and $\vec{b}$. 
Two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a} \cdot \vec{b} = 0$.

Ex.: Let $\vec{a} = \langle -1, 2, 5 \rangle$ and $\vec{b} = \langle 2, 2, 7 \rangle$, and find a non zero vector orthogonal to $\vec{b}$. Should you be able to find a vector orthogonal to both $\vec{a}$ and $\vec{b}$?
Projection

The projection of $\vec{b}$ onto $\vec{a}$ is

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}.$$ 

The component of $\vec{b}$ in the $\vec{a}$ direction is

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}.$$ 

Ex.: Let $\vec{a} = \ll -1, 2, 5 \gg$ and $\vec{b} = \ll 2, 2, 7 \gg$, and find the component of $\vec{b}$ in the $\vec{a}$ direction and the projection of $\vec{b}$ onto $\vec{a}$. 

LECTURE OUTLINE Dot product and cross product – p.8/1
Examples

Let \( \vec{a} = \langle -1, 2, 5 \rangle \) and \( \vec{b} = \langle 2, 2, 7 \rangle \). Find a length 3 vector such that its component in the \( \vec{b} \) is 2. What is your vectors component in the \( \vec{a} \) direction? Is it possible to find a length 3 vector such that its component in the \( \vec{b} \) is 2 which is perpendicular to \( \vec{a} \)?
Cross Product

Given vectors $\vec{a}$ and $\vec{b}$ we define $\vec{a} \times \vec{b}$ to be the unique vector satisfying

1. $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and to $\vec{b}$ (or zero).
2. It has length equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.
3. $\vec{a} \times \vec{b}$ is in the direction determined by the right hand rule going from $\vec{a}$ to $\vec{b}$. 


Main Theorem

Let \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \) and \( \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \), then \( \mathbf{a} \times \mathbf{b} \) equals

\[
(a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}.
\]

Example: Let \( \mathbf{a} = \langle -1, 2, 5 \rangle \) and \( \mathbf{b} = \langle 2, 2, 7 \rangle \) and find \( \mathbf{a} \times \mathbf{b} \). Find a length 3 vector such that its component in the \( \mathbf{b} \) is 2 which is perpendicular to \( \mathbf{a} \).