Ergodicity of quantum eigenfunctions in classically chaotic systems

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Classical billiards

Point particle in 2D domain \( \Omega \), elastic reflections off boundary \( \Gamma \).

Position \( \mathbf{r} \equiv (x, y) \). Phase space \((\mathbf{r}, \theta)\). Energy is conserved.

- Type of motion depends on billiard table shape:

Regular: has other conserved quantities (e.g. \( \theta \))
Classical billiards

Point particle in 2D domain $\Omega$, elastic reflections off boundary $\Gamma$. Position $\mathbf{r} \equiv (x, y)$. Phase space $(\mathbf{r}, \theta)$. Energy is conserved.

- Type of motion depends on billiard table shape:
  - Regular: has other conserved quantities (e.g. $\theta$)
  - Chaotic: ergodic: nearly every trajectory covers phase space

- Hyperbolicity: exponential divergence of nearby trajectories
  \[ |\mathbf{r}_1(t) - \mathbf{r}_2(t)| \sim e^{\Lambda t}, \quad \Lambda = \text{Lyapunov} \]
- Also: Anosov property (all $\Lambda > 0$), mixing (phase space flow)
‘Quantum’ billiards

Membrane (drum) problem: eigenfunctions $\phi_n(r)$ of laplacian

$$-\Delta \phi_n = E_n \phi_n, \quad \phi_n(r \in \Gamma) = 0 \quad \int_{\Omega} \phi_n^2 \, dr = 1$$

‘energy’ eigenvalue $E$

wavenumber $k$

$$k \equiv \sqrt{E} \equiv \frac{2\pi}{\lambda}$$

... cavity modes, quantum levels

‘quantized’ equivalent of classical billiards (momentum $\rightarrow i \nabla$)
Quantum chaos

What happens at higher $E$? Depends on classical dynamics:

Regular: $\phi_n$ separable

Chaotic: $\phi_n$ disordered

‘low’ energy: $n \sim 700$, $E \sim 10^4$, $k \sim 100$, $15\lambda$ across

1970’s to present day, field of QUANTUM CHAOS:

- eigenvalues (spacings, correlations, RMT…)
- eigenfunctions (ergodicity, correlations, matrix els…)
- dynamics (scattering, resonances, dissipation, electron physics, q. chemistry…)

(MOVIE)
Classical and quantum averages

Choose ‘test function’ \( A(\mathbf{r}) \):

**classical** (phase space) average
\[
\bar{A} \equiv \frac{1}{\text{vol}(\Omega)} \int_{\Omega} A(\mathbf{r}) d\mathbf{r}
\]

**quantum** version is \( \hat{A} = \text{operator in linear space of } \phi_n \)’s

Expectation (average)
\[
\langle \phi_n, \hat{A} \phi_n \rangle \equiv \int_{\Omega} A(\mathbf{r}) \phi_n(\mathbf{r})^2 d\mu_{\phi_n}
\]

Quantum ergodicity: \( h_n; \hat{A}^n \to 0 \) as \( E_n \to 1 \)

Does this happen? For all states \( A \)? At what rate?

We test numerically for certain \( A \), up to very high \( n \) \( \approx 10^6 \).

If true for all \( A \), equidistribution in space, \( d\mu_{\phi_n} \) uniform
Classical and quantum averages

Choose ‘test function’ \( A(r) \):

- **classical** (phase space) average \( \bar{A} \equiv \frac{1}{\text{vol}(\Omega)} \int_{\Omega} A(r) \, dr \)
- **quantum** version is \( \hat{A} = \) operator in linear space of \( \phi_n \)'s

Expectation (average) \( \langle \phi_n, \hat{A} \phi_n \rangle \equiv \int_{\Omega} A(r) \phi_n(r)^2 \, dr \)

Quantum ergodicity:

\[
\langle \phi_n, \hat{A} \phi_n \rangle - \bar{A} \to 0 \quad \text{as} \quad E_n \to \infty
\]

- Does this happen? For all states \( n \)? At what rate?
- We test numerically for certain \( A \), up to very high \( n \sim 10^6 \).

If true for all \( A \) \( \Rightarrow \) equidistribution in space, \( d\mu_{\phi_n} \to \text{uniform} \)
Outline

- Motivation: random waves, scars
- Ergodicity theorems, conjectures
- Numerical test results
- Rate of equidistribution: semiclassical estimate
- Sketch of numerical techniques which make this possible
- Conclusion
Motivation: Random plane waves

Conjecture (Berry ’77): statistical model of eigenfunctions

\[ \phi_n \sim \frac{1}{\sqrt{N}} \sum_{j=1}^{N} a_j \sin(k_j \cdot r + \alpha_j) \]

iid amplitudes \( a_j \in \mathbb{R} \)

iid phases \( \alpha_j \in [0, 2\pi[ \)

Wavevectors \( k_j \), spaced uniformly in direction, \( |k_j| = k \).

- Ray analogue of classical ergodicity.
- Predicts equidistribution as \( E \to \infty \):

\[ \text{deviations die like} \quad \left| \langle \phi_n, \hat{A}\phi_n \rangle - \bar{A} \right| \sim E^{-1/2} \]
High-energy eigenfunction $\phi_n$

$k \approx 10^3$
$E \approx 10^6$
$n \approx 5 \times 10^4$
Random plane waves

Stringy structures appear due to $|k| = \text{const.}$

Interesting…to the eye only?
Motivation: ‘Scars’

Heller ’84 observed: often mass concentrates (localizes) on short classical **unstable periodic orbits** (UPOs)…

Theory (Heller, Kaplan): on UPO higher classical return prob.

- Strong scars were thought to persist as $E \rightarrow \infty$. (No longer!)
- For certain $A$, our $\langle \phi_n, \hat{A}\phi_n \rangle$ is a measure of scarring
Quantum Ergodicity Theorem

QET (Schnirelman ’74, Colin de Verdière ’85, Zelditch ’87...):
For ergodic systems and well-behaved $A$,

$$\lim_{E_n \to \infty} \langle \phi_n, \hat{A}\phi_n \rangle - \bar{A} = 0$$

is true for almost all $\phi_n$.

- Could exist an exceptional set (scars?) which are not ergodic
- This set has to be a vanishing fraction of the total number

QET makes physicists happy: ‘Correspondence Principle’
all quantum & classical answers agree as $\lambda \to 0$
Quantum Unique Ergodicity

QUE conjecture (Rudnick & Sarnak ’94)
For every single eigenfunction,

$$\lim_{E_n \to \infty} \langle \phi_n, \hat{A} \phi_n \rangle - \bar{A} = 0$$

All converge to unique measure: $d\mu_\phi = \text{uniform. (no scars)}$
- Contrast classical flow has many invariant measures: each UPO

QUE was in context of hyperbolic manifolds…

Constant-curvature arithmetic case: recent analytic progress…
- Lindenstrauss ’03: measure can’t collapse on to UPO
- Luo & Sarnak ’03: bounds on sums $\Rightarrow$ deviations $\sim E^{-1/4}$

negative curvature
causes chaos
Numerical tests

Analytics only for special systems (symmetries, all $\Lambda = 1$)

Test generic chaotic system ($\Lambda$’s differ)

e.g. Sinai-type billiard: concave walls ⇒ Anosov

$A(r) = \text{piecewise const: fast quantum calc using boundary}$

- classical $\bar{A} = \frac{\text{vol}(\Omega_A)}{\text{vol}(\Omega)}$
- quantum $\langle \phi_n, \hat{A}\phi_n \rangle = \text{‘probability mass’ inside } \Omega_A$
Results: Expectation values

mean \( \langle \phi_n, \hat{A}\phi_n \rangle \rightarrow \bar{A} \).

Variance slowly decreasing, but how?
Results: Equidistribution rate

Quantum variance:

$$V_A(E) \equiv \frac{1}{m} \sum_{N \leq n < N+m} \left| \langle \phi_n, \hat{A}\phi_n \rangle - \bar{A} \right|^2$$

- Hard to measure: *e.g.* 1% needs $m \sim 2 \times 10^4$ indep samples!

Graph showing a power law $V_A(E) = aE^{-\gamma}$

- Fitted $\gamma = 0.48 \pm 0.01$

- ~25000 quantum states
**Results: Power law**

Variance $V_A(E) = aE^{-\gamma}$, found $\gamma = 0.48 \pm 0.01$

Consistent with conjecture that deviations $\sim E^{-1/4}$ (i.e. $\gamma = 1/2$)

Previous experiments also used piecewise-constant $A(r)$:

- Aurich & Taglieber ’98: negatively-curved surfaces, lowest $n < 6000$ only
- Bäcker ’98: billiards, $n < 6000$, but many choices of $A$

Can see power-law not asymptotic until $n \gtrsim 10^4$

\[ \ldots \text{we go 100 times higher!} \]

up to level $n \approx 8 \times 10^5$, $E \approx 1.6 \times 10^7$
Results: Distribution of deviations

Plot deviations scaled by $\sqrt{V_A(E)}$:

- Consistent with Gaussian (i.e. random wave model), convincing
Theory: Semiclassical variance estimate I

Feingold & Peres ’86

Signal $A(t) = \text{follow } A \text{ along particle trajectory } r(t)$

Consider autocorrelation of this signal:

$$\langle A(t)A(t + \tau) \rangle_t \overset{\text{ergod}}{=} \frac{A(0)A(\tau)}{QET} \approx \langle \phi_n, \hat{A}(0)\hat{A}(\tau)\phi_n \rangle$$

noise power spectrum

$\tilde{C}_A(\omega)$

$$\omega \equiv \text{distance from diagonal}$$

Barnett et al. ’00: verified in stadium billiard

- Note: the diagonal is our quantum expectation!
Theory: Semiclassical variance estimate II

Estimate $\tilde{C}_A(\omega)$ numerically:

Measure power spectrum of $A(t)$ along long trajectories

Physics: $\tilde{C}_A(\omega)$ is heating (dissipation) rate under external driving by field $A$.

(Cohen ’99: fluctuation-dissipation)
Theory: Semiclassical variance estimate II

Estimate \( \tilde{C}_A(\omega) \) numerically:

Measure power spectrum of \( A(t) \) along long trajectories

Physics: \( \tilde{C}_A(\omega) \) is heating (dissipation) rate under external driving by field \( A \).

DC limit \( \omega \to 0 \) gives diagonal variance:

\[
V_A(E) \equiv \text{var}(A_{nn}) \rightarrow \frac{2}{\text{vol}(\Omega)} \tilde{C}_A(\omega = 0) E^{-1/2}, \quad \leftarrow \gamma = 1/2
\]

Time-reversal symmetry: for diagonal, extra factor 2

(Cohen ’99: fluctuation-dissipation)
Results: semiclassical estimate

Power law: $V_A(E) = a E^{-\gamma}$

fitted $\gamma = 0.48 \pm 0.01$

Good agreement, no fitted params. (Estimate $a$ is 8% too big)

- Compare arithmetic surfaces: $a_{\text{classical}} \neq a_{\text{quantum}}$ provably
Numerical methods sketch

1. Compute eigenfunctions $\phi_n$ via **scaling method**:
   (Vergini & Saraceno ’94; correct explanation (QET) Barnett, Cohen & Heller ’00)
   
   **If:** find $A$ s.t. $\bar{A} \neq 0$ but dynamics gives $\lim_{\omega \to 0} \tilde{C}_A(\omega) = 0$
   
   **Then:** matrix $A_{nm} \approx$ diagonal: Eigenvectors of $A \approx \{\phi_n\}$.

   Put into a basis, size $N \sim 1/\lambda \sim \sqrt{E}$.  (e.g. $N \sim 4000$)
   
   - One dense matrix diagonalization returns $O(N)$ cluster of $\phi_n$’s
   - $O(N) \sim 10^3$ faster than boundary integral equation methods!
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2. New BASIS SETS of $Y_0$ Bessels placed outside the domain
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3. Norm formula for Helmholtz solutions (little known?):
   $$\langle \phi, \phi \rangle_{\Omega_A} = \frac{1}{2k^2} \int_{\partial \Omega_A} (n \cdot \nabla \phi)(r \cdot \nabla \phi) - \phi n \cdot \nabla (r \cdot \nabla \phi) \, ds$$
   • Overall effort scales $O(N^2)$ per state (few CPU-days total)
Missing levels?

Weyl’s estimate for $N(E)$, the # eigenvalues $E_n < E$:

$$N_{\text{Weyl}}(E) = \frac{\text{vol}(\Omega)}{4\pi} E - \frac{L}{4\pi} \sqrt{E} + O(1) \cdots$$

- not 1 state missing in sequence of 6812 states
Conclusion

- Are quantum (laplacian) eigenfunctions spatially uniform in chaotic systems as $E \to \infty$?
  - Measured rate of equidistribution in generic billiard
  - Unprecedented range in $E$ & sample size
  - Strong support for QUE conjecture (no scars)
  - Power law consistent with conjectured $\gamma = 1/2$
  - Semiclassical estimate good, not perfect

- Directions
  - Study prefactor $a$ for other choices of $A$ (does it vary?)
  - Variant of QUE: off-diagonal matrix elements?
  - Scaling method: basis sets, rigor, fast Bessels...