Ergodicity of quantum eigenfunctions in classically chaotic systems

Mar 1, 2004

Alex Barnett
barnett@cims.nyu.edu

Courant Institute

work in collaboration with Peter Sarnak, Courant/Princeton
Classical billiards

Point particle in 2D domain $\Omega$, elastic reflections off boundary $\Gamma$. Position $\mathbf{r} \equiv (x, y)$. Phase space $(\mathbf{r}, \theta)$. Energy is conserved.

- Type of motion depends on billiard table shape:

Regular: has other conserved quantities (e.g. $\theta$)
Classical billiards

Point particle in 2D domain $\Omega$, elastic reflections off boundary $\Gamma$. Position $\mathbf{r} \equiv (x, y)$. Phase space $(\mathbf{r}, \theta)$. Energy is conserved.

- Type of motion depends on billiard table shape:
  - Regular: has other conserved quantities (e.g. $\theta$)
  - Chaotic: ergodic: nearly every trajectory covers phase space

- Hyperbolicity: exponential divergence of nearby trajectories
  \[ |\mathbf{r}_1(t) - \mathbf{r}_2(t)| \sim e^{\Lambda t}, \quad \Lambda = \text{Lyapunov} \]
- Also: Anosov property (all $\Lambda > 0$), mixing (phase space flow)
‘Quantum’ billiards

Membrane (drum) problem: eigenfunctions $\phi_n(r)$ of laplacian

$$-\Delta \phi_n = E_n \phi_n, \quad \phi_n(r \in \Gamma) = 0 \quad \int_{\Omega} \phi_n^2 \, dr = 1$$

‘energy’ eigenvalue $E$

wavenumber $k$

$$k \equiv \sqrt{E} \equiv \frac{2\pi}{\lambda}$$

... cavity modes, quantum levels

‘quantized’ equivalent of classical billiards (momentum $\rightarrow i\nabla$)
Quantum chaos

What happens at higher $E$? Depends on classical dynamics:

Regular: $\phi_n$ separable

Chaotic: $\phi_n$ disordered

‘low’ energy: $n \sim 700$, $E \sim 10^4$, $k \sim 100$, $15\lambda$ across

1970’s to present day, field of QUANTUM CHAOS:

- eigenvalues ( spacings, correlations, RMT . . .)
- eigenfunctions ( ergodicity, correlations, matrix els . . . )
- dynamics ( scattering, resonances, dissipation, electron physics, q. chemistry . . . )
Classical and quantum averages

Choose ‘test function’ $A(r)$:

**classical** (phase space) average $\bar{A} \equiv \frac{1}{\text{vol}(\Omega)} \int_{\Omega} A(r) \, dr$

**quantum** version is $\hat{A} = \text{operator in linear space of } \phi_n$’s

Expectation (average) $\langle \phi_n, \hat{A} \phi_n \rangle \equiv \int_{\Omega} A(r) \phi_n(r)^2 \, dr$

$d\mu_{\phi_n}$ density measure
Classical and quantum averages

Choose ‘test function’ \( A(\mathbf{r}) \):

*classical* (phase space) average \( \overline{A} \equiv \frac{1}{\text{vol}(\Omega)} \int_{\Omega} A(\mathbf{r}) d\mathbf{r} \)

*quantum* version is \( \hat{A} = \text{operator in linear space of } \phi_n \)'s

Expectation (average) \( \langle \phi_n, \hat{A} \phi_n \rangle \equiv \int_{\Omega} A(\mathbf{r}) \phi_n(\mathbf{r})^2 d\mathbf{r} \)

**Quantum ergodicity:**

\[
\langle \phi_n, \hat{A} \phi_n \rangle - \overline{A} \longrightarrow 0 \quad \text{as } E_n \rightarrow \infty
\]

- Does this happen? For all states \( n \)? At what *rate*?
- We test numerically for certain \( A \), up to very high \( n \sim 10^6 \).

If true for all \( A \) \( \Rightarrow \) equidistribution in space, \( d\mu_{\phi_n} \rightarrow \text{uniform} \)
Outline

- Motivation: random waves, scars
- Ergodicity theorems, conjectures
- Numerical test results
- Rate of equidistribution: semiclassical estimate
- Sketch of numerical techniques which make this possible
- Conclusion
Motivation: Random plane waves

Conjecture (Berry ’77): statistical model of eigenfunctions

\[ \phi_n \sim \frac{1}{\sqrt{N}} \sum_{j=1}^{N} a_j \sin(k_j \cdot r + \alpha_j) \]

\text{iid amplitudes } a_j \in \mathbb{R} \\
\text{iid phases } \alpha_j \in [0, 2\pi[ \\

Wavevectors \( k_j \), spaced uniformly in direction, \( |k_j| = k \).

- Ray analogue of classical ergodicity.
- Predicts equidistribution as \( E \to \infty \):

\[ \text{deviations die like } \left| \langle \phi_n, \hat{A} \phi_n \rangle - \overline{A} \right| \sim E^{-1/4} \]
High-energy eigenfunction $\phi_n$

$k \approx 10^3$
$E \approx 10^6$
$n \approx 5 \times 10^4$
Random plane waves

stringy structures appear due to $|k| = \text{const.}$

Interesting ... to the eye only?
Motivation: ‘Scars’

Heller ’84 observed: often mass concentrates (localizes) on short classical unstable periodic orbits (UPOs)...

Theory (Heller, Kaplan): on UPO higher classical return prob.

- Strong scars were thought to persist as $E \to \infty$. (No longer!)
- For certain $A$, our $\langle \phi_n, \hat{A}\phi_n \rangle$ is a measure of scarring
Quantum Ergodicity Theorem

QET (Schnirelman ’74, Colin de Verdière ’85, Zelditch ’87...):

For ergodic systems and well-behaved $A$,

$$\lim_{E_n \to \infty} \langle \phi_n, \hat{A} \phi_n \rangle - \overline{A} = 0$$

is true for almost all $\phi_n$.

- Could exist an exceptional set (scars?) which are not ergodic
- This set has to be a vanishing fraction of the total number

QET makes physicists happy: ‘Correspondence Principle’

all quantum & classical answers agree as $\lambda \to 0$
Quantum Unique Ergodicity

QUE conjecture (Rudnick & Sarnak ’94)
For every single eigenfunction,

$$\lim_{E_n \to \infty} \langle \phi_n, \hat{A} \phi_n \rangle - \bar{A} = 0$$

All converge to unique measure: \(d\mu_\phi = \text{uniform. (no scars)}\)
- cf. classical flow which has many invariant measures: each UPO

QUE was in context of hyperbolic manifolds…

Constant-curvature arithmetic case: recent analytic progress…
- Lindenstrauss ’03: measure can’t collapse on to UPO
- Luo & Sarnak ’03: bounds on sums \(\Rightarrow\) deviations \(\sim E^{-1/4}\)
Numerical tests

Analytics only for special systems (symmetries, all $\Lambda = 1$)

Test **generic** chaotic system ($\Lambda$’s differ)

e.g. Sinai-type billiard: concave walls $\Rightarrow$ Anosov

$A(r) = \text{piecewise const}: \text{fast quantum calc using boundary}$

- classical $\bar{A} = \frac{\text{vol}(\Omega_A)}{\text{vol}(\Omega)}$
- quantum $\langle \phi_n, \hat{A}\phi_n \rangle = \text{‘probability mass’ inside } \Omega_A$
Results: Expectation values

\[ \langle \phi_n, \hat{A}\phi_n \rangle \rightarrow \overline{A}. \]

Variance slowly decreasing, but how?
Results: Equidistribution rate

Quantum variance: \( V_A(E) \equiv \frac{1}{m} \sum_{N \leq n < N+m} \left| \langle \phi_n, \hat{A} \phi_n \rangle - \bar{A} \right|^2 \)

- Hard to measure: *e.g.* 1% needs \( m \sim 2 \times 10^4 \) indep samples!

\[ V_A(E) \approx a E^{-\gamma} \]

\( \gamma \) fitted: \( \gamma = 0.48 \pm 0.01 \)

\( \sim 25000 \) quantum states
Results: Power law

Variance $V_A(E) = aE^{-\gamma}$, found $\gamma = 0.48 \pm 0.01$

Consistent with conjecture that deviations $\sim E^{-1/4}$ (i.e. $\gamma = 1/2$)

Previous experiments also used piecewise-constant $A(r)$:

- Aurich & Taglieber ’98: negatively-curved surfaces, lowest $n < 6000$ only
- Bäcker ’98: billiards, $n < 6000$, but many choices of $A$

Can see power-law not asymptotic until $n \gtrsim 10^4$

...we go 100 times higher!

up to level $n \approx 8 \times 10^5$, $E \approx 1.6 \times 10^7$
Results: Distribution of deviations

Histogram deviations after scaling by $\sqrt{V_A(E)}$:

- Consistent with Gaussian (*i.e.* random wave model), convincing
Theory: Semiclassical variance estimate I

Signal $A(t) =$ follow $A$ along particle trajectory $r(t)$

Consider autocorrelation of this signal:

$$\langle A(t)A(t + \tau) \rangle_t \stackrel{\text{ergod}}{=} \overline{A(0)A(\tau)} \stackrel{\text{QET}}{\approx} \langle \phi_n, \hat{A}(0)\hat{A}(\tau)\phi_n \rangle$$
Theory: Semiclassical variance estimate I

Feingold & Peres ’86

Signal $A(t) = \text{follow } A$ along particle trajectory $r(t)$
Consider autocorrelation of this signal:

$$\langle A(t)A(t + \tau) \rangle_t \overset{\text{ergod}}{=} \frac{A(0)A(\tau)}{QET} \approx \langle \phi_n, \hat{A}(0)\hat{A}(\tau)\phi_n \rangle$$

↓

fourier transform ↓

noise power spectrum

$\tilde{C}_A(\omega)$

$\omega \equiv \text{distance from diagonal}$

Barnett et al. ’00: verified in stadium billiard

• Note: the diagonal is our quantum expectation!
Theory: Semiclassical variance estimate II

Estimate $\tilde{C}_A(\omega)$ numerically:

Measure power spectrum of $A(t)$ along long trajectories

Physics: $\tilde{C}_A(\omega)$ is heating (dissipation) rate under external driving by field $A$.

(Cohen ’99: fluctuation-dissipation)
Theory: Semiclassical variance estimate II

Estimate $\tilde{C}_A(\omega)$ numerically:

Measure power spectrum of $A(t)$ along long trajectories

Physics: $\tilde{C}_A(\omega)$ is heating (dissipation) rate under external driving by field $A$.

DC limit $\omega \rightarrow 0$ gives diagonal variance:

$$V_A(E) \equiv \text{var}(A_{nn}) \rightarrow \frac{2}{\text{vol}(\Omega)} \tilde{C}_A(\omega = 0) E^{-1/2}, \quad \leftarrow \gamma = 1/2$$

Time-reversal symmetry: extra factor $\text{var}(A_{nn}) \approx 2\text{var}(A_{nm})$
Results: Semiclassical estimate

Power law: $V_A(E) = a E^{-\gamma}$

fitted $\gamma = 0.48 \pm 0.01$

No fitted params. Good agreement, but estimate $a$ 5\% too big

- cf. arithmetic surfaces where $a_{\text{classical}} \neq a_{\text{quantum}}$ provably
Most accurate test ever for real system, no fitted params.

- Error at $\omega = 0$ related to QM peak exaggeration?
Numerical methods sketch

1. Compute eigenfunctions $\phi_n$ via **scaling method**:
   
   (Vergini & Saraceno ’94; correct explanation (QET) Barnett, Cohen & Heller ’00)

   **If:** find $A$ s.t. $\overline{A} \neq 0$ but dynamics gives $\lim_{\omega \to 0} \tilde{C}_A(\omega) = 0$
   
   **Then:** matrix $A_{nm} \approx$ diagonal: Eigenvectors of $A \approx \{\phi_n\}$.

   Put into a basis, size $N \sim 1/\lambda \sim \sqrt{E}$. (e.g. $N \sim 4000$)
   
   - One dense matrix diagonalization returns $O(N)$ cluster of $\phi_n$'s
   - $O(N) \sim 10^3$ faster than boundary integral equation methods!
Numerical methods sketch

1. Compute eigenfunctions $\phi_n$ via **scaling method**:
   (Vergini & Saraceno ’94; correct explanation (QET) Barnett, Cohen & Heller ’00)

   **If:** find $A$ s.t. $\overline{A} \neq 0$ but dynamics gives $\lim_{\omega \to 0} \tilde{C}_A(\omega) = 0$

   **Then:** matrix $A_{nm} \approx$ diagonal: Eigenvectors of $A \approx \{\phi_n\}$.

   Put into a basis, size $N \sim 1/\lambda \sim \sqrt{E}$. (e.g. $N \sim 4000$)

   - One dense matrix diagonalization returns $O(N)$ cluster of $\phi_n$’s
   - $O(N) \sim 10^3$ faster than boundary integral equation methods!

2. New **BASIS SETS** of $Y_0$ Bessels placed outside the domain
Numerical methods sketch

1. Compute eigenfunctions $\phi_n$ via scaling method:
   (Vergini & Saraceno ’94; correct explanation (QET) Barnett, Cohen & Heller ’00)

   If: find $A$ s.t. $\overline{A} \neq 0$ but dynamics gives $\lim_{\omega \to 0} \tilde{C}_A(\omega) = 0$

   Then: matrix $A_{nm} \approx$ diagonal: Eigenvectors of $A \approx \{\phi_n\}$.

   Put into a basis, size $N \sim 1/\lambda \sim \sqrt{E}$.  (e.g. $N \sim 4000$)
   • One dense matrix diagonalization returns $O(N)$ cluster of $\phi_n$’s
   • $O(N) \sim 10^3$ faster than boundary integral equation methods!

2. New BASIS SETS of $Y_0$ Bessels placed outside the domain

3. Norm formula for Helmholtz solutions (little known?):

   $\langle \phi, \phi \rangle_{\Omega_A} = \frac{1}{2k^2} \int_{\partial\Omega_A} \left( n \cdot \nabla \phi \right) \left( r \cdot \nabla \phi \right) - \phi n \cdot \nabla \left( r \cdot \nabla \phi \right) \, ds$

   • Overall effort scales $O(N^2)$ per state (few CPU-days total)
Missing levels?

Weyl’s estimate for $N(E)$, the # eigenvalues $E_n < E$:

$$N_{\text{Weyl}}(E) = \frac{\text{vol}(\Omega)}{4\pi} E - \frac{L}{4\pi} \sqrt{E} + O(1) \cdots$$

- not 1 state missing in sequence of 6812 states
Conclusion

- Are quantum (laplacian) eigenfunctions spatially uniform in chaotic systems as $E \to \infty$?
  - Measured rate of equidistribution in generic billiard
  - Unprecedented range in $E$ & sample size
  - Strong support for QUE conjecture (all scars vanish)
  - Power law consistent with conjectured $\gamma = 1/2$
  - Semiclassical estimate good, not perfect

- Directions
  - Prefactor $a$: agreement in semiclassical limit?
  - How about for other choices of $A$ (error varies?)
  - Variant of QUE: off-diagonal matrix elements?
  - Scaling method: basis sets, rigor, other shapes...