1 Simple Harmonic Oscillator (SHO)

Paradigm system is mass $m$ attached to spring strength $k$. Motion in 1d is scalar function $x(t)$. We have 1 degree of freedom (dof).

- Newton’s 2nd Law: $m \ddot{x} = F$
- Hooke’s Law (linear response): $F = -kx$, where $x = 0$ is the equilibrium position.

Gives linear, 2nd-order ordinary differential equation (ODE), $\ddot{x} + \frac{k}{m}x = 0$.

Substituting $x(t) = ae^{i\omega_0 t}$ we find that only two roots $\omega = \pm \omega_0$ are possible if $a \neq 0$, where $\omega_0 = \pm \sqrt{k/m}$. Any linear combination of these two rotating complex exponentials is a solution. There are 3 common equivalent ways to write the general solution,

$$x(t) = \text{Re} \left[ ae^{i\omega_0 t} \right] = A \cos \omega_0 t + B \sin \omega_0 t = C \sin(\omega_0 t + \phi), \quad (1)$$

in a way that restricts to real (physical) solutions. $a \in \mathbb{C}$ and $A, B, C, \phi \in \mathbb{R}$. This is called harmonic motion. Motion is periodic with period $T$, related to the angular frequency $\omega_0$ by $\omega_0 = 2\pi f = 2\pi / T$.

$C$ is amplitude (also $a$ can be called a [complex] amplitude), and $\phi$ is phase. Note smaller mass or more rigid spring both lead to higher frequency.

An initial value problem consists of finding the coefficients in a general solution which match a given position and velocity at e.g. time $t = 0$.

2 Normal Modes

A system described by $N$ coordinates $\mathbf{x} \equiv \{x_1, x_2, \ldots, x_N\}$ has $N$ dofs. A normal mode is a solution where all coordinates move with the same period:

$$x_i(t) = a_i e^{i\omega t}. \quad (2)$$

Note: $i = \sqrt{-1}$ in exponential $\neq$ subscript $i$ labeling coordinate.

Motion is separable into a product of a function $a_i$ depending only on coordinate $i$ and a function $e^{i\omega t}$ depending only on time $t$. 

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Assume same mass for each coord, so $m \ddot{x}_i = F_i$

Most general linear response is $F_i = -\sum_{j=1}^{N} K_{ij} x_j$, where $K$ is an $N \times N$ matrix, generally symmetric ($K = K^T$).

Gives (coupled) ODE, in vector notation: $\ddot{x} + \frac{1}{m} K x = 0$.

Substituting Eq.(2) gives, $(K - m\omega^2 I)a = 0$.

Nontrivial ($a \neq 0$) solution only for discrete mode frequencies $\omega_n = +\sqrt{\frac{\lambda_n}{m}}$ where $\lambda_n$, $n = 1 \ldots N$ are the eigenvalues (also called spectrum) of $K$.

Each mode $n$ must then have amplitude $a_n$ proportional to an eigenvector $v_n$ of $K$ corresponding to eigenvalue $\lambda_n$.

- Linear algebra reminder: eigenvalue equation $Kv_n = \lambda_n v_n$.
- $K$ symm $\Leftrightarrow$ real $\lambda_n$, real orthogonal $v_n$.
- Choose unit-length eigenvectors so orthonormal, $v_m \cdot v_n = \delta_{mn}$
- Kronecker delta defined by $\delta_{mn} = 1$ if $m = n$, 0 if $m \neq n$.
- Note we ignored issues arising if $\lambda$'s not distinct.

General solution is a linear combination of periodic normal modes but is itself generally not periodic (because the mode frequencies need not have rational ratios). General solution for each mode is as in Eq.(1), giving, in vector form,

$$x(t) = \text{Re} \left[ \sum_{n=1}^{N} a_n e^{i\omega_n t} v_n \right] = \sum_{n=1}^{N} C_n \sin(\omega_n t + \phi_n) v_n.$$  \hspace*{1cm} (3)

Initial value problem: using the first form in Eq.(3), we want set $\{a_n\}$ given initial position $x(0)$, velocity $\dot{x}(0)$. We find,

$$a_m = \text{real part } + \text{ imag part } = v_m \cdot x(0) - i \frac{1}{\omega_m} v_m \cdot \dot{x}(0) \hspace*{1cm} (4)$$

Deriving this involved taking dot product of $v_m$ with Eq.(3), and using the useful rule $\sum_m \delta_{mn} f_m = f_n$ where $f$ is any function on the integers.

### 2.1 2-mass linear chain

In class we discussed the case of two masses $m$ connected together and to fixed walls by three springs $k$, moving along a line. There are $N = 2$ dofs: $x_1, x_2$ are the horizontal displacement of each mass from equilibrium.

We showed that the spring matrix was $K = k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Dy diagonalizing $K$, we find the two modes are:
• \( \lambda_1 = k \) giving \( \omega_1 = +\sqrt{k/m} \) and \( \mathbf{v}_1 = (1, 1) \), so the masses move similarly.
• \( \lambda_2 = 3k \) giving \( \omega_2 = +\sqrt{3k/m} \) and \( \mathbf{v}_2 = (1, -1) \), so the masses move opposite to each other.

Any general motion is a combination of these 2 modes, and we emphasize that it is generally not periodic.
See the course webpage for link to an applet which allows you to play with this system.