We will show that the spectrum of a compact hyperbolic 2-orbifold determines the number and type of its cone-points, and the combined length of its reflector boundary. The proof is a routine application of Selberg’s formula, together with some simple hyperbolic geometry.

Once we’ve read off from the spectrum the data about cone-points and reflectors, we can go on to show that the spectrum also determines the remaining ‘Selbergian parameters’, namely, the lengths and orientation classes of its closed geodesics. This extension is a small step, in the sense that it follows from the known case of not-necessarily-orientable hyperbolic surfaces. This extension is at the same time something of a giant leap, since showing that two isospectral not-necessarily-orientable hyperbolic surfaces have geodesics of the same lengths and types is a delicate business.

YIKES! I forgot the contributions to the spectrum coming from glide reflections along boundary reflectors. These are homophonic with translations along orientation-reversing geodesics, and thus with translations along orientation-preserving geodesics. So who knows what subtle isospectralities might be possible for hyperbolic 2-orbifolds? In this connection, it should be pointed out there could already have been isospectralities where boundary corners match with interior cone-points. Should this count already as subtle isospectrality?

Denote the contribution to the counting trace attributable to an orbifold point of order \( n \) by \( g_n(t) \). This contribution is the sum of contributions attributable to the \( n-1 \) non-trivial classes of curves encircling the orbifold point. When \( n = 2k + 1 \) is odd, these classes match up in pairs, going opposite ways around around the cone point:

\[
g_{2k+1}(t) = \sum_{i=1}^{k} 2g_{i,2k+1}(t).
\]

When \( n = 2k \) is even, there is one oddball class, where the geodesics head straight in to the cone point and ‘bounce’ back:

\[
g_{2k}(t) = g_{k,2k}(t) + \sum_{i=1}^{k-1} 2g_{i,2k}(t).
\]

**Proposition 1.**

\[
g_{i,n}(t) = \frac{1}{n} f_{\frac{2\pi i}{n}}(t),
\]

\[
1
\]
where

\[ f_\theta(t) = 2\pi \left( \frac{1 + \sinh^2 \frac{t}{2}}{\sin^2 \frac{\theta}{2}} - 1 \right). \]

**Proof.** From the lore of the counting trace,

\[ g_{i,n}(t) = \frac{1}{n} f_{2m_i}(t), \]

where \( f_\theta(t) \) is the area \( 2\pi(\cosh r - 1) \) of a hyperbolic disk of such a radius \( r \) that a chord of length \( t \) subtends angle \( \theta \). We just have to make sure that we have the formula for \( f_\theta \) right. The quantities \( r, \theta, t \) satisfy

\[ \sinh r = \frac{\sin \frac{t}{2}}{\sin \frac{\theta}{2}}. \]

Thus

\[
\begin{align*}
f_\theta(t) &= 2\pi(\cosh r - 1) \\
&= 2\pi \left( \sqrt{1 + \sinh^2 r} - 1 \right) \\
&= 2\pi \left( \sqrt{1 + \frac{\sinh^2 \frac{t}{2}}{\sin^2 \frac{\theta}{2}}} - 1 \right).
\end{align*}
\]

Hopefully it will seem absolutely incredible that there might be any non-trivial linear relation between these functions \( f_\theta \). There isn’t:

**Proposition 2.** The functions \( f_\theta, 0 < \theta \leq \pi \) are linearly independent: No non-trivial linear combination of any finite subset of these functions vanishes identically on any interval \( 0 < t < T \).

**Proof.** It suffices to show linear independence of the family of functions \( \sqrt{1 + au} - 1, 1 < a < \infty \), on all intervals \( 0 < u < U \). (Set \( u = \sinh^2 \frac{t}{2} \) and \( a = \frac{1}{\sin^2 \frac{\theta}{2}} \).) Actually functions of this form remain independent when \( a \) is allowed to range throughout the complex plane, and not just over the specified real interval. A quick way to see this is to write in Taylor series

\[
g(u) = \sqrt{1 + u} - 1 = b_1 u + \frac{b_2}{2} u^2 + \frac{b_3}{6} u^3 + \ldots
\]
and observe (look up?) that only the constant term vanishes:

\[ b_1, b_2, \ldots \neq 0. \]

If a linear combination \( \sum_{i=1}^{n} c_i g(a_i u) \) is to vanish on \( 0 < u < U \), then all its derivatives must vanish at \( u = 0 \):

\[ \sum_{i=1}^{n} c_i a_i^k b_k = 0, \quad k = 1, 2, \ldots. \]

(We ignore the constant term because all the functions involved here vanish at \( u = 0 \).) Dividing by \( b_k \) gives

\[ \sum_{i=1}^{n} c_i a_i^k = 0, \quad k = 1, 2, \ldots. \]

Could this system of equations for \( c_1, \ldots, c_n \) have a non-trivial solution? If so, then the subsystem consisting of only the first \( n \) of these equations would have a non-trivial solution:

\[ \sum_{i=1}^{n} c_i a_i^k = 0, \quad k = 1, \ldots, n. \]

Since the row-rank and column-rank of a matrix are the same, this would mean that the adjoint system

\[ \sum_{k=1}^{n} a_i^k d_k = 0, \quad i = 1, 2, \ldots, n \]

would also have a non-trivial solution. But this would mean that the non-zero degree-\( n \) polynomial

\[ P(X) = \sum_{k=1}^{n} d_k X^k \]

would have \( n + 1 \) distinct roots \( X = 0 \) and \( X = a_1, \ldots, a_n \), which is impossible.

This proof is too slick, in that we are relying on the fact that none of the derivatives of \( \sqrt{1 + u} \) happen to vanish at \( u = 0 \). The proof could be made to work even if some of these derivatives did vanish. It would just be a bit more complicated. We’ve chosen to simplify the discussion by seizing upon an accidental feature of the problem. That’s a cheat, and we invite you to think about how to prove this fact in an honest and above-board fashion.

Now for the reflectors. Denote by \( R \) the combined length of all reflecting boundaries.
Proposition 3. The combined contribution of the reflecting boundaries to the counting trace is

\[ R \sinh \frac{t}{2}. \]

Proof. Straight-forward.

Proposition 4. We can read off from the spectrum the combined length \( R \) of the reflecting boundaries, and the number and type of all orbifold points.

Proof. The spectrum determines the counting trace. The only contributions to the counting trace other than those from cone points and reflectors come from closed geodesics, and these don’t kick in until \( t \) exceeds the length of the shortest closed geodesic. The reflectors make a first-order contribution to the counting trace at \( u = 0 \), and the cone-points only a second-order contribution, so we can read off \( R \) from the first derivative of the counting trace at \( u = 0 \). We subtract out the contribution of the reflectors from the counting trace. Then we proceed in the usual way, taking advantage of the linear independence of the functions \( f_\theta \): Determine the highest order of cone-points that are present, and the number of these; subtract out their contributions, and proceed by induction.

Proposition 5. We can read off all relevant Selbergian parameters from the spectrum.

Proof. Subtract from the counting trace the contributions of the reflectors and cone-points. What remains arises from the closed geodesics. In the manifold case, it is shown that the characteristics of the closed geodesics are determined by the counting trace, and hence by the spectrum. The proof carries over here essentially without change.