Tetra and Didi, the cosmic spectral twins

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Abstract

We introduce a pair of isospectral but non-isometric compact flat 3-manifolds named Tetra (a tetracosm) and Didi (a didicosm). The closed geodesics of Tetra and Didi are very different. Where Tetra has two quarter-twisting geodesics of the shortest length, Didi has four half-twisting geodesics. Nevertheless, the Selberg trace formula shows these spaces to be isospectral, as the result of a delicate interplay between the lengths and twists of their closed geodesics. This isospectrality can also be proven directly by matching eigenfunctions having the same eigenvalue.

Introducing Tetra and Didi

A platycosm is a compact flat 3-manifold. Simplest among platycosms are the torocosms (the artifacts formerly known as ‘3-dimensional tori’). Torocosms come in various shapes and sizes. Among these, we distinguish the cubical torocosm $\mathbb{R}^3/\mathbb{Z}^3$, and the two-story torocosm TwoTall $= \mathbb{R}^3/((\mathbb{Z} \times \mathbb{Z}) \times 2\mathbb{Z})$.

All other platycosms arise as quotients of torocosms. There are 10 distinct types in all, of which 6 (torocosm; dicosm; tricosm; tetracosm; hexacosm; didicosm) are orientable. The spaces themselves are well known, but the naming scheme, due to Conway, is new. The naming scheme and the spaces themselves will be described in great detail in a forthcoming work by Conway and Rossetti [2]. The spaces are described under different names by Weeks [14], and Weeks (see [15]) has also produced software which allows you to ‘fly around’ inside these spaces, and many others as well.

Here we are concerned with two specific platycosms: Tetra, a tetracosm, and Didi, a didicosm. Please note that the prefix ‘didi-’ is a doubling of
the prefix ‘di-’, and not some exotic Greek root. The word ‘didicosm’ is pronounced ‘die-die-cosm’, but Didi is pronounced ‘Dee-dee’. Tetra and Didi turn out to be, up to scale, the unique pair of *cosmic spectral twins* (non-isometric platycosms with identical Laplace spectrum).

Tetra and Didi are both 4-fold quotients of TwoTall. Tetra is the quotient of TwoTall by a fixed-point-free action of $\mathbb{Z}/4\mathbb{Z}$, while Didi is the quotient by a fixed-point-free action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. To get Tetra, we adjoin to the translation group $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ in $(x, y, z)$-space the quarter-turn screw motion

$$\tau : (x, y, z) \mapsto (-y, x, z + 1/2).$$

To get Didi, we adjoin instead the two half-turn screw motions

$$\rho_x : (x, y, z) \mapsto (x + 1/2, -y, -z)$$

and

$$\rho_y : (x, y, z) \mapsto (-x, y + 1/2, 1 - z),$$

which together with the translations generate a third half-turn screw motion

$$\rho_z : (x, y, z) \mapsto (1/2 - x, 1/2 - y, z + 1).$$

Both Tetra and Didi have as a fundamental domain the box

$$[-1/2, 1/2] \times [-1/2, 1/2] \times [0, 1/2],$$

and in both cases the four vertical sides are glued up in parallel in the usual way, front to back and left to right, yielding a stack of square tori. The difference comes in the gluings of the top and bottom. (See Figure 1.) To get Tetra, you use $\tau$ to glue the bottom to the top with a quarter-turn. To get Didi, you use $\rho_x$ to glue the bottom to itself via a glide reflection, and $\rho_y$ to glue the top to itself via a glide reflection. These gluings produce two Klein bottles embedded in Didi. There is also a third, ‘vertical’ Klein bottle, associated to $\rho_z$.

**Note.** We have described Tetra and Didi as quotients of a common 4-fold cover. In fact they have a common 2-fold cover, the ‘cubical dicosm’, and a common 2-fold orbifold quotient. Neither will play a role in our discussion.
Figure 1: Tetra and Didi. The sides of the box glue back to front and left to right in the usual way; the tops and bottoms glue as indicated. Note that in the case of Didi, the top and bottom glue not to each other but each to itself, yielding two Klein bottles embedded in the quotient (which is nonetheless orientable!).

Non-isometric

Tetra and Didi are not isometric. In fact, since they have different fundamental groups, they are not even homeomorphic. Moreover, in contrast to many of the known examples of spectral twins, their closed geodesics are markedly different. (See Figure 2.)

In Tetra there are two quarter-twisting geodesics of length $1/2$, one running up the middle of the box along the line $x = y = 0$, and one running up the four identified edges of the box. The vertical midlines of the four sides of the box combine to give a third geodesic, but this one is a half-twisting geodesic of length 1.

In Didi, there are four half-twisting geodesics of length $1/2$, two associated with $\rho_x$ sitting in the Klein bottle gotten by gluing the bottom of the box, and two associated with $\rho_y$ sitting in the Klein bottle gotten by gluing the top. (See Figure 2.) In addition, there are two half-twisting geodesics sitting in the $\rho_z$ Klein bottle, but these have length 1.

Let’s call a geodesic twisted if it has a non-trivial twist. We’ve identified 3 twisted geodesics in Tetra, and 6 in Didi. These are all primitive, which means that they don’t arise by going more than once around a shorter geodesic. In fact these are the only primitive twisted geodesics in Tetra and Didi. Of course there are also imprimitive twisted geodesics, which come from going around a half-twister an odd number of times, or around a quarter-twister a number of times not divisible by 4.
To see that we have identified all the twisted geodesics, note that any twisted geodesic in Tetra (or Didi) unwraps to a straight line in $\mathbb{R}^3$. Translating-with-a-twist along this straight line will be a ‘covering transformation’—that is, one of the symmetries of $\mathbb{R}^3$ consistent with all patterns obtained by unwrapping patterns in the quotient Tetra (or Didi). The translation is by the length of the geodesic, and the twist is equal (and opposite) to the twist of the geodesic. In the case of Tetra, the line must be vertical, because all covering translations-with-a-nontrivial-twist run vertically. In the case of Didi, the line can run in any of the three coordinate directions. Look carefully at the possibilities, and you’ll see that in listing nontrivially-twisting geodesics we’ve accounted for all of them.

In this census of closed geodesics, we are counting each pair of oppositely-oriented geodesics only once. Because of the duplicity of geodesics, there is a real possibility that some of our formulas below will be off by a factor of 2. However, it will be clear that our arguments are robust enough that this will not be a matter of concern. (That is, until we come to discuss non-orientable hyperbolic surfaces.)
Isospectral

While Tetra and Didi are not isometric, they are isospectral. By definition, two spaces are isospectral if there exists some way of matching up the eigenfunctions of the Laplacian of the two spaces so that corresponding eigenfunctions have the same eigenvalue. In this section, we will show that Tetra and Didi are isospectral by describing such a correspondence.

We are giving this explicit proof because it is entirely elementary—it relies only on linear algebra and Fourier series—and because it is illuminating in its own way. Other, more ‘conceptual’ proofs are available. Further along, we will outline one such proof, by way of the Selberg trace formula. A third proof can be obtained using the general machinery developed by Miatello and Rossetti in [8], and a fourth using the ‘dual’ approach of [9]. A close relative of this fourth proof emerges naturally in the proof that Tetra and Didi are the unique pair of cosmic spectral twins [3], discussed briefly below.

A function on Tetra corresponds to a function \( f \) on TwoTall that is invariant under \( \tau \), in that

\[ f = f \circ \tau. \]

Given any function \( f \) on TwoTall, we can symmetrize under \( \tau \) to get a \( \tau \)-invariant function

\[ \sigma^{\text{Tetra}}(f) = \frac{1}{4}(f + f \circ \tau + f \circ \tau \circ \tau + f \circ \tau \circ \tau \circ \tau), \]

which we think of as a function on Tetra. Similarly, we can get functions on Didi via the symmetrization

\[ \sigma^{\text{Didi}}(f) = \frac{1}{4}(f + f \circ \rho_x + f \circ \rho_y + f \circ \rho_z). \]

Now any function \( f \) on TwoTall can be written as a Fourier series:

\[ f(x, y, z) = \sum_{(a, b, c) \in \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2}\mathbb{Z}} \hat{f}(a, b, c) \exp(2\pi i(ax + by + cz)). \]

The Fourier basis functions \( \phi_{a,b,c} = \exp(2\pi i(ax + by + cz)) \), \((a, b, c) \in \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2}\mathbb{Z}\) are eigenfunctions of the (positive) Laplacian \( \Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \):

\[ \Delta \phi_{a,b,c} = 4\pi^2(a^2 + b^2 + c^2)\phi_{a,b,c}. \]

Symmetrizing these Fourier basis functions under \( \tau \) yields a spanning set \( \sigma^{\text{Tetra}}(\phi_{a,b,c}) \) for the functions on Tetra. This spanning set is far from being
a basis. For one thing, symmetrization lumps the basis functions together in groups, generally of size four. More important, symmetrizing a basis function can kill it off altogether. For example,

\[ \sigma^{\text{Tetra}}(\phi_{0,0,1/2}) = \sigma^{\text{Tetra}}(\phi_{0,0,1}) = \sigma^{\text{Tetra}}(\phi_{0,0,3/2}) = 0. \]

However, by eliminating such redundancies, we can prune down to a basis of (unnormalized) eigenfunctions \( \sigma^{\text{Tetra}}(\phi_{a_i,b_i,c_i}) \) on Tetra, with corresponding eigenvalues \( 4\pi^2(a_i^2 + b_i^2 + c_i^2) \).

Similarly, we can get a basis of eigenfunctions \( \sigma^{\text{Didi}}(\phi_{a'_i,b'_i,c'_i}) \) on Didi, with corresponding eigenvalues \( 4\pi^2(a'_i^2 + b'_i^2 + c'_i^2) \). If we can arrange that

\[ a_i^2 + b_i^2 + c_i^2 = a'_i^2 + b'_i^2 + c'_i^2 \]

for all \( i \), then we will have verified that Tetra and Didi are spectral twins.

To find such a correspondence, we will take advantage of the fact that our two symmetrization mappings lump the basis functions together in two different but very nearly compatible ways. This leads to a correspondence between eigenfunctions which is for the most part very straight-forward (though, as we will observe, not quite ‘canonical’). There are only two exceptional cases that must be treated carefully.

Let

\[ V_{a,b,c} = \langle \phi_{\pm a, \pm b, \pm c}, \phi_{\pm b, \pm a, \pm c} \rangle. \]

(Please observe that \( V_{a,b,c} = V_{b,a,c}; V_{a,b,c} = V_{-a,b,c}; \text{ etc.} \)) In the generic case, namely when \( a, b, c \neq 0 \) and \( |a| \neq |b| \), the vector space \( V_{a,b,c} \) is 16-dimensional, and both \( \sigma^{\text{Tetra}} \) and \( \sigma^{\text{Didi}} \) lump the 16 basis functions together in groups of 4. In this case \( \sigma^{\text{Tetra}}(V_{a,b,c}) \) and \( \sigma^{\text{Didi}}(V_{a,b,c}) \) are both 4-dimensional, and we can clearly take bases of these spaces and match them up.

If it were true that \( \dim \sigma^{\text{Tetra}}(V_{a,b,c}) = \dim \sigma^{\text{Didi}}(V_{a,b,c}) \), for all \( (a, b, c) \in \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2}\mathbb{Z} \), we would be all set. In fact this equality holds as long as no two of the parameters \( a, b, c \) vanish, because in these cases \( \sigma^{\text{Tetra}} \) and \( \sigma^{\text{Didi}} \) continue to lump the basis functions together in groups of 4. Of course \( \dim \sigma^{\text{Tetra}}(V_{0,0,0}) = \dim \sigma^{\text{Didi}}(V_{0,0,0}) = 1 \), so that case is no problem. And if \( c \) is a half integer, then \( \dim \sigma^{\text{Tetra}}(V_{0,0,c}) = \dim \sigma^{\text{Didi}}(V_{0,0,c}) = 0 \).

So the question comes down to how to handle the cases \( V_{n,0,0} \) and \( V_{0,0,n} \), with \( n \) a non-zero integer, which we may assume is positive. (Remember that negating any of \( a, b, c \) does not change the space \( V_{a,b,c} \).) To extend the correspondence between eigenfunctions, we must take these remaining exceptional cases in combination. Here is how it goes.
Odd exceptional case. When \( n \) is a positive odd integer,

\[
\begin{align*}
\sigma_{\text{Tetra}}(V_{n,0,0}) &= \langle \cos 2\pi nx + \cos 2\pi ny \rangle; \\
\sigma_{\text{Tetra}}(V_{0,0,n}) &= 0; \\
\sigma_{\text{Didi}}(V_{n,0,0}) &= 0; \\
\sigma_{\text{Didi}}(V_{0,0,n}) &= \langle \cos 2\pi nz \rangle.
\end{align*}
\]

Taken together, these cases contribute a single eigenfunction of eigenvalue \( 4\pi^2 \cdot n^2 \) to the spectra of both Tetra and Didi.

Even exceptional case. When \( n \) is a positive even integer,

\[
\begin{align*}
\sigma_{\text{Tetra}}(V_{n,0,0}) &= \langle \cos 2\pi nx + \cos 2\pi ny \rangle; \\
\sigma_{\text{Tetra}}(V_{0,0,n}) &= \langle \exp(2\pi inz), \exp(-2\pi inz) \rangle; \\
\sigma_{\text{Didi}}(V_{n,0,0}) &= \langle \cos 2\pi nx, \cos 2\pi ny \rangle; \\
\sigma_{\text{Didi}}(V_{0,0,n}) &= \langle \cos 2\pi nz \rangle.
\end{align*}
\]

Taken together, these cases contribute three independent eigenfunctions of eigenvalue \( 4\pi^2 \cdot n^2 \) to both spectra.

By matching up these exceptional cases as indicated, we can finish the job of matching up eigenfunctions of Tetra and Didi, and thus concretely demonstrate that these spaces are spectral twins.

Note that any specific scheme for matching eigenfunctions involves some arbitrary choices. This shows up clearly in even exceptional case above, but it is an issue even in the generic case, because there is no way to avoid breaking symmetry when choosing a correspondence between eigenfunctions. The lack of a canonical matching reflects the fact that the isospectrality of these two spaces is a subtle business. We will see this again when we look at the proof of isospectrality by way of the Selberg trace formula.

Unique

Tetra and Didi are, up to scale, the only pair of non-isometric isospectral platycosms: They are the two-and-only cosmic spectral twins. The proof, due to Rossetti, involves a case-by-case analysis of all possible spectral coincidences among and between platycosms of the 10 possible types. In [3], Conway and Rossetti give a streamlined version of this proof, using Conway’s theory of lattice conorms as an organizing principle. As you would expect,
the techniques used in proving uniqueness yield another proof that Tetra and Didi are spectral twins.

A key ingredient in the uniqueness proof is Schiemann’s theorem [12] that there are no spectral twins among torocoms: If \( \mathbb{R}^3/\Lambda_1 \) and \( \mathbb{R}^3/\Lambda_2 \) are isospectral, then they (and the lattices \( \Lambda_1 \) and \( \Lambda_2 \)) are isometric. Milnor’s original example of spectral twins was a pair of 16-dimensional tori [10]. Subsequently, lower-dimensional pairs of isospectral tori were found, culminating with the discovery of a 4-dimensional pair by Schiemann, simplified and extended to a 4-parameter family of pairs by Conway and Sloane [4]. Schiemann showed that as far as tori are concerned, dimension 4 is the end of the line. By opening the field up to other flat manifolds, we can get down to dimension 3—but just barely!

**Selberg**

Here, as promised above, we outline a proof of isospectrality by way of the Selberg trace formula. The version of the trace formula that we want to use expresses the Laplace transform of the spectrum (or properly speaking, the spectral measure) as the sum of contributions attributable to families of closed geodesics. To show that Tetra and Didi are isospectral, we will examine the closed geodesics of each, and check that the total spectral contribution of Tetra’s geodesics is just the same as that of Didi’s.

The relevant computations are indicated in Table 1. Here we will explain informally what lies behind the computations in the table. The discussion is contrived in such a way as to allow us to put off actually writing down Selberg’s formula until after we have put it to use.

Our reason for preferring this inverted approach is that the Selberg formula is easier to apply than to state (in the present case, at least). Certain features of the formula are relevant to its application, while others are not. Once readers have seen how the formula is applied, we think they will be better able to appreciate the salient features of the explicit formula. And perhaps, if we explain clearly enough how the formula is applied, readers may feel no need to mess with the explicit formula at all!

Recall that when it comes to the shortest geodesics, which have length 1/2, Tetra has two ‘quarter-twisters’, while Didi has four ‘half-twisters’. Now it happens that, in a flat 3-manifold, the spectral contribution of any primitive quarter-twisting geodesic is just twice that of a primitive half-twisting geodesic. (This is an aspect of a general phenomenon: ‘The more the twist;
Table 1: Balancing geodesics. This table shows the balancing of the spectral contributions from the nontrivially twisting geodesics in Tetra and Didi. Here \( l \) is length, and \( w_l \) the total spectral contribution (weight) of geodesics of length \( l \), measured in units of the spectral contribution of an imprimitive half-twisting geodesic of length \( l \). The point of this table is to demonstrate that \( w_l \) is the same for Tetra and Didi. For geodesics of a specific kind, \( n \) tells the number of geodesics; \( t \) the twist (either \( \frac{1}{4} \) or \( \frac{1}{2} \)); \( k \) the imprimitivity exponent; and \( w \) the aggregate spectral weight for geodesics of this kind.

An individual geodesic with imprimitivity exponent \( k \) has weight \( \frac{1}{k} \) if it is half-twisting, and \( 2/k \) if it is quarter-twisting. Weights do not depend on the handedness of the twist, so we do not distinguish between \( 1/4 \)-twisting and \( 3/4 \)-twisting geodesics.

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So as far as the shortest geodesics go, the contributions to the spectrum are the same.

Next come geodesics of length 1. Both Tetra and Didi have two 2-dimensional families of non-twisting geodesics, which they inherit from the common cover TwoTall. These common families of non-twisting geodesics make identical contributions to the spectrum. In general, the non-twisting geodesics in Tetra and Didi are all inherited from the common cover, and consequently contribute equally to their spectra. So we don’t have to worry about non-twisting geodesics.

Looking at nontrivially twisting geodesics of length 1, the only kind that arise are half-twisters. In Tetra we already identified one primitive half-twister running vertically up the midlines of the sides of the box we have chosen as our fundamental domain; in Didi, we have two primitive half-twisters sitting in the vertical Klein bottle. That’s it, as far as primitive geodesics are concerned. However, in Tetra, we also have two imprimitive half-twisters, gotten by running twice around those two primitive quarter-twisters of length 1/2. In the Selberg formula the spectral contribution of an imprimitive geodesic must be divided by its degree of imprimitivity or exponent, which is the number of times it runs around its primitive ancestor. So the spectral contribution of Tetra’s one new half-twister and two recycled quarter-twisters just matches that of Didi’s two brand new half-twisters.

Next among nontrivially twisting geodesics are those of length 3/2. Here we are back to balancing Tetra’s two quarter-twisters, now thrice-imprimitive, against Didi’s four half-twisters, also thrice-imprimitive.

At length 2, there are no non-twisting geodesics.

Length 5/2 is like 1/2 and 3/2: Tetra has two quarter-twisters and Didi four half-twisters, all now five-times-imprimitive.

Length 3 is like length 1: We are back to balancing Didi’s two half-twisters, now thrice-imprimitive, against Tetra’s one half-twister, now thrice-imprimitive, and two quarter-twisters, now recycled as six-times-imprimitive half-twisters.

And so it goes on up the line. Thus Tetra and Didi are isospectral.

**Formula**

We have chosen to describe the geodesic balancing act between Tetra and Didi in words, rather than symbols, in order to put off having to state explicitly Selberg’s formula for platycosms. But now the time has come.
Let $M = \mathbb{R}^3/\Gamma$ be a platycosm with covering group $\Gamma$, and let $\Lambda \subset \Gamma$ be the lattice subgroup of $\Gamma$. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of the Laplacian on $M$. For a geodesic $g$, let $l(g)$ be its length, $\theta(g)$ its twist (in radians), and $k(g)$ its imprimitivity exponent. Let $G$ be the set of nontrivially-twisting (unoriented) closed geodesics of $M$.

Here is the version of the trace formula we need.

$$\sum_n e^{-\lambda_n t} = \int_0^\infty \frac{1}{(2\pi t)^\frac{3}{2}} e^{-\frac{s^2}{4t}} dN(s),$$

where

$$N(s) = \text{Vol}(M) \{|\lambda| \leq s\} + 2 \sum_{g \in G} \frac{1}{k(g)} V(l(g), \theta(g), s),$$

where

$$V(h, \theta, s) = l(g) \pi \frac{s^2 - h^2}{(2 \sin \frac{\theta}{2})}$$

if $s \geq h$ and 0 otherwise. Here $V(h, \theta, s)$ is the volume of a cylinder of height $h$ and $\theta$-twisted height $s$: If the sides of a cylinder of height $h$ are replaced with parallel rubber bands, and the top is twisted through an angle $\theta$, the bands stretch to form (part of) a hyperboloid of one sheet; their stretched length is what we’re calling the $\theta$-twisted height.

The great thing about this formula is that the function $\sum_n e^{-\lambda_n t}$ determines the spectrum: In fact, it is the Laplace transform of a mass distribution with a unit mass placed at every eigenvalue, so we can recover the spectrum by inverting a Laplace transform. The trace formula thus shows that the Laplace spectrum is determined by the function $N(s)$, which is computed directly from geometrical data.

The use we have made above of the trace formula depends on three observations. First is the amazing fact that the contribution of a nontrivially-twisting geodesic to $N(s)$ depends on the twist $\theta$ only through the factor $\frac{1}{(2 \sin \frac{\theta}{2})}$. For a half-twisting geodesic ($\theta = \pi$), this factor is 1; for a quarter-twisting geodesic ($\theta = \pi/2$), this factor is 2. Second is the way recycled geodesics get only partial credit in $N(s)$, because of the factor $\frac{1}{k(g)}$. Third is the fact that for our purposes, it does not matter if the formula is off by a constant factor here or there, say a stray $\sqrt{2\pi}$, or a factor of 2 arising through confusion about oriented versus unoriented geodesics. As long as the error applies equally to Tetra and Didi, it does us no harm.
About orienting those geodesics: In the formula for $N(s)$, the factor of 2 in front of the sum over geodesics is supposed to account for the two possible orientations; there is no factor of 2 in front of the first term of $N(s)$, since each family of untwisted geodesics is represented twice in $\Lambda$. So hopefully the formula is correct as far as this goes.

The version of the Selberg formula stated here can be gleaned from the papers of Gangolli [5] and Berard-Bergery [1]. Doyle [?] gives a leisurely account, emphasizing that the geometrical tools needed to derive the formula come down to the Pythagorean theorem and the volume of a cylinder. A similar approach to Selberg’s formula works for hyperbolic surfaces and 3-manifolds, with the hyperbolic law of cosines taking the place of the Pythagorean theorem.

The trace formula for platycosms can also be derived by adapting the classical Poisson summation formula, which was the original inspiration for the Selberg formula; This is the approach of Sunada [13] and Miatello and Rossetti [9].

Subtle isospectrality

The remarkable Selbergian interplay between the geodesics in Tetra and Didi is what we were looking for when we discovered this pair. Originally, we were interested in finding (or ruling out) an analogous pair of hyperbolic 3-manifolds. There are plenty of examples of spectral twins among hyperbolic 3-manifolds, but the standard methods for producing spectral twins yield pairs whose geodesics have matching lengths and twists. Selberg’s formula seems to allow the possibility of twins that are subtly isospectral. After some fruitless attempts to find such a pair among hyperbolic 3-manifolds, we tried looking among flat 3-manifolds instead. The case of hyperbolic 3-manifolds remains open.

Remarks

Misconception about Selberg for hyperbolic 3-manifolds Some people mistakenly believe that, for a hyperbolic 3-manifold, Selberg’s formula allows you to read off from the spectrum the lengths and twists of the closed geodesics. For example, Reid [11] cites Gangolli [5] and Berard-Bergery [1] in support of this assertion. Neither Gangolli nor Berard-Bergery makes such a
statement, and Gangolli in particular is explicit about the fact that the Selberg formula leaves open the possibility of an example of the kind we were (and still are) looking for.

This misconception most likely stems from conflicting uses of the term length spectrum, which we have been studiously avoiding here.

**Hyperbolic surfaces** In another attempt to work up to hyperbolic 3-manifolds, we looked at hyperbolic surfaces. We were disappointed to find that no subtly isospectral pairs exist among hyperbolic surfaces. The question only becomes interesting in the case of non-orientable surfaces, because the Selberg formula immediately implies that for hyperbolic surfaces, spectral twins always have matching lengths. For non-orientable surfaces, it is possible to construct a plausible scenario for matching the contributions of the geodesics of a pair of surfaces in a way similar to that of Tetra and Didi, where geodesics don’t have matching lengths. But in the end it proves impossible to make this work: We show that to balance the spectral contributions of geodesics on a pair of surfaces whose geodesics can’t be matched so as to preserve length and orientability class, you would need the number of geodesics with length $\leq l$ to be asymptotically at least $2e^l/l$; but by results of Huber [6, 7] and others, this number must be asymptotically exactly $e^l/l$, which is a factor of 2 short of what we need. (Huber’s results are stated only for orientable surfaces, but they hold as well in the non-orientable case.)

Thus, while it seems like we come tantalizingly close to being able to build a pair of hyperbolic surfaces that are subtly isospectral, in the end we are forced to conclude that any pair of spectral twins among hyperbolic surfaces must have geodesics with matching lengths and twists.

**Note.** The fact that we are off by a factor of 2 suggests that perhaps we simply dropped a factor of 2 somewhere—say, by forgetting that a geodesic can be oriented in 2 ways. Rossetti says he is sure that we haven’t messed up in this way, and in any case the factor by which we’re off is greater than 2. Doyle is less confident. One thing Rossetti and Doyle agree on is that it is frustrating to have to rely on such a delicate argument. This delicate argument may well be unnecessary. All this ‘tantalizingly close’ stuff could just be nonsense. Our scenario for building an example involves having many geodesics of exactly the same length. Maybe this could be ruled out summarily, without some delicate argument about overall growth rate for geodesics of a given maximum length.
**Lengths and multiplicities**  For a manifold of negative curvature, it is clear how to count the multiplicity of geodesics of a given length. (Well, almost clear: There is always a question of whether to count oppositely-oriented geodesics as the same or different.) For flat manifolds, where closed geodesics can come in continuous families, there is room for disagreement about the correct way to measure multiplicity. But however you decide to measure multiplicity, you are going to conclude that Tetra and Didi have different multiplicities of geodesics of the shortest length, because these shortest geodesics are all isolated.

**Laplacian on forms**  While Tetra and Didi are isospectral for the usual Laplacian acting on functions, they are not isospectral for the Laplacian acting on 1-forms or 2-forms. This is a simple consequence of the techniques of Miatello and Rossetti (see Theorem 3.1 of [8]). Examples of manifolds isospectral on functions but not on 1-forms were previously known only in higher dimensions.

**References**


