Differentiation Rules

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Differentiability is Stronger than Continuity

Theorem. If $f'(a)$ exists, then $f$ is continuous at $a$.

A function whose derivative exists at every point of an interval is not only continuous, it is smooth, i.e. it has no sharp corners.
Building the Toolbox

**Theorem.** Suppose $y = f(x)$ is a function that has derivative $f'$. Then, $(cf)' = cf'$, where $c$ is a constant. Or in Leibniz’s notation

$$
\frac{d}{dx}(cf(x)) = c\Delta \frac{d}{dx}f(x).
$$
**Theorem.** If \( f \) and \( g \) are functions with derivatives \( f' \) and \( g' \), respectively, then \((f + g)' = f' + g'\). In words, the derivative of a sum is the sum of the derivatives.
Examples

\[
\frac{d}{dx}(3x^2 + 2x + 7)
\]

\[
\frac{d}{dx}(x + \sqrt{x})
\]
The Product Rule

**Theorem.** If \( f \) and \( g \) are functions with derivatives \( f' \) and \( g' \), respectively, then \((fg)' = fg' + gf'\). In words, “the derivative of a product is the first factor times the derivative of the second, plus the second factor times the derivative of the first”.
Examples

• Find \( f'(x) \) in two ways, given \( f(x) = (5x + 3)(x + 2) \).

• If \( y = \sqrt{x}(x^2 + 2) \), find \( \frac{dy}{dx} \).
The Reciprocal of Calculus Modeling

**Theorem.** Suppose $f$ has derivative $f'$. Then for any $x$ such that $f(x) \neq 0$, \( \left( \frac{1}{f} \right)' = -\frac{f'(x)}{f(x)^2} \). That is, \( \left( \frac{1}{f} \right)' = -\frac{f'}{f^2} \).
Example

• Find $f'(x)$ given $f(x) = \frac{1}{x^2 + 1}$.
The Quotient Rule

**Theorem.** Suppose $f$ and $g$ have derivatives $f'$ and $g'$, respectively. Then for any $x$ such that $g(x) \neq 0$, \( \left( \frac{f}{g} \right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \). That is, \( \left( \frac{f}{g} \right)' = \frac{gf' - fg'}{g^2} \). In words, “the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the denominator squared”. 

Building the Toolbox ...
Examples

• Find $f'(x)$ given

$$f(x) = \frac{x + 1}{x + 2}.$$ 

• Find $f'(x)$ given

$$f(x) = \frac{1 + \sqrt{x}}{x^2 + 3x + 2}.$$
Example

• For \( f(x) = \frac{1}{x} = x^{-1} \), find the derivative three ways, using the power rule, the reciprocal rule, and the quotient rule.
The Chain Rule

**Theorem.** Let \((f \circ g)(x) = f(g(x))\) be the function defined from \(f\) and \(g\) by composition. Assume that \(g\) is differentiable at the point \(x\) and that \(f\) is differentiable at the point \(g(x)\). Then the composite function \(f \circ g\) is differentiable at the point \(x\), and

\[
(f \circ g)'(x) = \left[ f(g(x)) \right]' = f'(g(x))g'(x)
\]

Using Leibniz’s notation:

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]
Examples

• Differentiate

\[ f(x) = \sqrt{x^2 + 1}. \]

• Differentiate

\[ y = (x^2 + 2)^{10}. \]
Examples

- Differentiate

\[ f(x) = (1 + 3\sqrt{x})^{35}. \]

- Differentiate

\[ f(x) = \left( \frac{x + 1}{x^2 + 1} \right)^3. \]