The Mean Value Theorem

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The Mean Value Theorem

**Theorem.** Suppose that $f$ is defined and continuous on a closed interval $[a, b]$, and suppose that $f'$ exists on the open interval $(a, b)$. Then there exists a point $c$ in $(a, b)$ such that

$$
\frac{f(b) - f(a)}{b - a} = f'(c).
$$
The Mean Value Theorem ...
The Mean Value Theorem ...

Examples

- Discontinuity at an end point
- Discontinuity at an interior point $p$
- No derivative at an interior point $p$
Examples ...

- Consider the function $f(x) = |x|$ on $[-1, 1]$.

- The Mean Value Theorem does not apply because the derivative is not defined at $x = 0$. 
• Under what circumstances does the Mean Value Theorem apply to the function $f(x) = 1/x$?
• Verify the conclusion of the Mean Value Theorem for the function \( f(x) = (x + 1)^3 - 1 \) on the interval \([-3, 1]\).
Recall

- An interval $I$ is the set of real numbers lying between $a$ and $b$, where $a$ and $b$ are real numbers or $\pm \infty$. 
**Definition**

Suppose that $f$ is defined on an interval $I$, and let $x_1$ and $x_2$ denote points in $I$:

1. $f$ is *increasing* on $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

2. $f$ is *decreasing* on $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

3. $f$ is *nondecreasing* on $I$ if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.

4. $f$ is *nonincreasing* on $I$ if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$. 
Theorem. Let $I$ be an interval and let $J$ be the open interval consisting of $I$ minus its endpoints (if any). Suppose that $f$ is continuous on $I$ and differentiable on $J$. Then

1. If $f'(x) > 0$ for every $x \in J$, then $f$ is increasing on $I$.

2. If $f'(x) < 0$ for every $x \in J$, then $f$ is decreasing on $I$.

3. If $f'(x) \geq 0$ for every $x \in J$, then $f$ is nondecreasing on $I$.

4. If $f'(x) \leq 0$ for every $x \in J$, then $f$ is nonincreasing on $I$. 
Examples

On what interval is the function $f(x) = x^3 + x + 1$ increasing (decreasing)?
Find the intervals on which the function \( f(x) = 2x^3 - 6x^2 - 18x + 1 \) is increasing and those on which it is decreasing.
Theorem. If \( f \) is continuous on a closed interval \([a, b]\), then there is a point \( c_1 \) in the interval where \( f \) assumes its maximum value, i.e. \( f(x) \leq f(c_1) \) for every \( x \) in \([a, b]\), and a point \( c_2 \) where \( f \) assumes its minimum value, i.e. \( f(x) \geq f(c_2) \) for every \( x \) in \([a, b]\).
**Theorem.** If $f$ is defined in an open interval $(a, b)$ and achieves a maximum (or minimum) value at a point $c \in (a, b)$ where $f'(c)$ exists, then $f'(c) = 0$. 
For the function $f(x) = 2x^3 - 6x^2 - 18x + 1$, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>53</td>
</tr>
<tr>
<td>$-4$</td>
<td>$-151$</td>
</tr>
<tr>
<td>4</td>
<td>$-39$</td>
</tr>
</tbody>
</table>
**Rolle’s Theorem**

**Theorem.** Suppose that the function $g$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. If $g(a) = 0$ and $g(b) = 0$ then there exists a point $c$ in the open interval $(a, b)$ where $g'(c) = 0$. 

![Diagram of Rolle's Theorem](image)
Use Rolle’s Theorem to show that the equation $x^5 - 3x + 1 = 0$ has exactly three real roots.
Implicit Differentiation

• Many curves are not the graphs of functions.

• A circle of radius 1, for example, does not pass the “vertical line test” and hence is not the graph of a function.

• It is, however, the graph of the equation $x^2 + y^2 = 1$. 
The equation \( x^3 - 8xy + y^3 = 1 \) resists our most clever efforts to explicitly solve for \( y \) as a function of \( x \).

We will see how to overcome this difficulty using a very important technique called implicit differentiation.
• The general setting for our discussion of implicitly defined functions is an equation $F(x, y) = 0$, where $F$ is an expression containing the two variables $x$ and $y$.

• A function $f(x)$ is said to be implicitly defined by the equation if $F(x, f(x)) = 0$ on some interval $I$.

• GOAL: Find the derivative of $f(x)$ without explicitly solving the equation.
Examples

• The functions $\sqrt{1 - x^2}$ and $-\sqrt{1 - x^2}$ are implicitly defined by the equation $x^2 + y^2 = 1$.

• Consider one of the functions $f(x)$ defined implicitly by the equation $x^2 + y^2 = 1$. Consider one of the functions $f(x)$ defined implicitly by the equation $x^2 + y^2 = 1$.

\[
f'(x) = -\frac{x}{f(x)}.\]
Examples ... 

- Given the equation $x^2 + y^2 = 1$, we think of the functions $y = f(x)$ implicitly defined by the equation.

\[
\frac{dy}{dx} = -\frac{x}{y}.
\]
• Use implicit differentiation to find the equation of the tangent line to the graph of $xy^2 + x^2y - 6 = 0$ at the point $(1, 2)$.  

Examples ...
Return to the equation $x^3 - 8xy + y^3 = 1$ with which we begin this section. Find the slope at the points on the curve for which $x = 1$. 

\[ x^3 - 8xy + y^3 = 1 \]
Examples ...

- Suppose a differentiable function $f$ has an inverse $f^{-1}$. Find the derivative of $f^{-1}$.

\[
\frac{dy}{dx} = \frac{1}{f'(y)}
\]

\[
[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}
\]