Slope Fields and Euler’s Method

11/02/2005
• We are going to study the geometric information that we get from a differential equation that gives an explicit formula for the derivative.

• Consider the differential equation

\[ \frac{dy}{dx} = F(x, y); y(x_0) = y_0. \]

• The equation says that at any point \((x, y)\) in the plane we can compute the slope \(\frac{dy}{dx}\) of the tangent line through that point.
• At each point \((x, y)\) in the plane, we can draw a short straight line whose slope is \(F(x, y)\) from the differential equation.

• The resulting two-dimensional plot of tangent lines is called the slope field or direction field of the differential equation.
• Let $F(x, y) = 8\sqrt{y}$

<table>
<thead>
<tr>
<th>Point $(x, y)$</th>
<th>Slope $F(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0)$</td>
<td>0</td>
</tr>
<tr>
<td>$(1, 1)$</td>
<td>8</td>
</tr>
<tr>
<td>$(1, 4)$</td>
<td>16</td>
</tr>
</tbody>
</table>
• Can you guess the shape of the solution curve that passes through (0, 0)?
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Example

- We will consider the differential equation \( \frac{dy}{dx} = 24x^3 \).
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Euler’s Method

• Assume that the following IVP is given:

\[
\frac{dy}{dx} = F(x, y); \quad P_0 = (x_0, y_0)
\]

• The method consists of starting at the initial point \( P_0 = (x_0, y_0) \), specifying an increment \( x \), and plotting a sequence of line segments joined end to end.

• The slope of each segment is the value of the derivative at the initial point of the segment.
slope = \( F(x_1, y_1) \)
slope = \( F(x_2, y_2) \)
slope = \( F(x_0, y_0) \)
Theorem. Given the Initial Value Problem

\[ \frac{dy}{dx} = F(x, y); \quad P0 = (x_0, y_0), \]

and \( \Delta x \) specified, then the endpoints of the line segments that make up the polygonal path in Euler’s Method are

\[ \begin{align*}
    x_{n+1} &= x_n + \Delta x \\
    y_{n+1} &= y_n + \Delta x F(x_n, y_n)
\end{align*} \]
Example

• Let $\frac{dy}{dx} = x - y; \ y(0) = 1$.

• On the interval $[0, 1]$ approximate $y(1)$ with two steps of size $1/2$.

• Here $F(x, y) = x - y$ and $\Delta x = 1/2$. 
Example

• Let \( \frac{dy}{dx} = x - y; \ y(0) = 1 \).

• On the interval \([0, 1]\) approximate \( y(1) \) with two steps of size \( 1/2 \).

• Here \( F(x, y) = x - y \) and \( \Delta x = 1/2 \).

• Thus, \( y_1 = y_0 + \Delta x F(x_0, y_0) = 1 + (1/2)(-1) = 1/2 \);

• \( y_2 = y_1 + \Delta x F(x_1, y_1) = 1/2 + (1/2)(0) = 1/2 \).

• Therefore \( y(1) \approx 1/2 \).
Case Study: Population Modeling

• **Objective:** Predict the size of the US population well into the 21st century.
• Translate real-world problems into mathematical models.

• Subject the models to mathematical analysis and prediction.

• Draw conclusions from the models.

• Test the conclusions in the laboratory and compare the results with the original real-world data.

• Revise the model as necessary and repeat the above steps until the model is a reliable predictor of real-world observations.
The Malthus and Verhulst Models

- The **Malthus** model for growth of a population assumes an ideal environment.

- Resources are unlimited, disease is constrained, and individuals are happy.

- The population increases at a rate proportional to the number of individuals present.
• The *Verhulst* model assumes that the growth rate declines, from a value $k$ when conditions are very favorable, to the value 0 when the population has increased to the maximum value $M$ that the environment can support.

• Takes into account the effects of a limiting environment.

• It is a more realistic model.
• We will use only the populations recorded in the census of 1790 and of 1990.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>3.9</td>
</tr>
<tr>
<td>1990</td>
<td>250</td>
</tr>
</tbody>
</table>
The Malthus Model: Exponential Growth

Starting with a population of 3.9 million in 1790, we have the Initial Value Problem

\[
\frac{dQ}{dt} = kQ; Q(90) = 3.9
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- Have fun with the applet!
The Verhulst Model: Limited Exponential Growth

- The Verhulst model assumes that the growth rate declines, from a value $k$ when conditions are very favorable, to the value 0 when the population has increased to the maximum value $M$ that the environment can support.

- It replaces the growth constant $k$ by the expression

$$k \frac{M - Q(t)}{M}$$
This leads to the differential equation

\[
\frac{dQ}{dt} = k \frac{M - Q}{M} Q.
\]
• The factor $\frac{M-Q}{M}$ that has a value between 0 and 1 is sometimes called the unrealized potential for population growth.

• When $Q$ is small it has a value close to 1, and the growth of the population is essentially exponential.

• As $Q$ approaches its asymptotic limiting value, however, the factor $\frac{M-Q}{M}$ is close to zero, and the population grows ever more slowly.
Objective

• The U.S. population cannot sustain exponential growth indefinitely.

• The Malthus model gives unrealistic projections of the population over the next century.

• We would like to use the Verhulst model instead to make such projections.

• We also need to assume that $Q(0) = 3.9$ million, and $M = 750$ million, the maximum value of the population ($0 \leq Q(t) \leq M$).
HAVE FUN!

• And don’t forget to use the applets!