Arc Length

11/15/2005
• Another illustration of the Riemann Sum modeling method, consider the problem of computing the length of a curve in the plane.

• We will assume that $y = f(x)$ is a continuous function defined on the interval $[a, b]$ and that $f'(x)$ exists at every point of the interval.

• How to determine the length of the graph of $f$ from the point $(a, f(a))$ to the point $(b, f(b))$. 
Summary of the Riemann Sum Method for Arc Length

- Divide the interval \([a, b]\) into \(n\) subintervals of equal length \(x = (b - a)/n\). Call the points of the subdivision \(a = x_0 \leq x_1 \leq x_2 \leq x_3 \Delta x \cdots \leq x_{n-1} \leq x_n = b\), where \(x_i = a + i\Delta x\) for each \(i\).

- On each subinterval \([x_{i-1}, x_i]\) connect the points \((x_{i-1}, f(x_{i-1}))\) and \((x_i, f(x_i))\) on the graph of \(f\) with straight lines.

- The length of the straight-line segment connecting the two points is
  \[(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2.\]
\[ y = f(x) \]

Points:
- \((a, f(a))\)
- \((b, f(b))\)

Intervals:
- \(a\)
- \(x_{i-1}\)
- \(x_i\)
- \(b\)
\[
\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}
\]

\[
\Delta s = \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right)} = \Delta x \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}
\]
Summary ...

• We can replace $\Delta s$ by the approximate value

$$\Delta s \simeq \Delta x \sqrt{1 + [f'(x_{i-1})]^2}.$$

• The sum of the approximate lengths of these line segments provides an approximation to the length of the curve

$$\sum_{i=1}^{n} \sqrt{1 + [f'(x_{i-1})]^2} \Delta x.$$

• Taking the limit as $x \to 0$, the above approximation approaches the length of the curve. The limit is

$$L = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(x_{i-1})]^2} \Delta x = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} dx.$$
The Arc Formula

• The integral formula to compute the length $L$ of the graph of $f$ between $x = a$ and $x = b$ is

$$L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.$$
Example

• Find the length of the arc $y = x^{3/2}$, from $x = 0$ to $x = 1$. 
Example

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\[
\begin{align*}
L &= \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} \, dx \\
&= \int_0^1 \sqrt{1 + \left(\frac{9}{4}x\right)} \, dx
\end{align*}
\]
• Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from $x = 1$ to $x = 2$. 
• Find the length of the curve \( y = x^4 + \frac{1}{32x^2} \) from \( x = 1 \) to \( x = 2 \).

\[
y' = 4x^3 - \frac{2}{32x^3} = 4x^3 - \frac{1}{16x^3}
\]

\[
L = \int_1^2 \sqrt{1 + \left(4x^3 - \frac{1}{16x^3}\right)^2} \, dx
\]

\[
= \int_1^2 \sqrt{1 + 16x^6 - \frac{8}{16} + \frac{1}{256x^6}} \, dx
\]

\[
= \int_1^2 \sqrt{\frac{8}{16} + 16x^6 + \frac{1}{256x^6}} \, dx
\]

\[
= \int_1^2 \sqrt{\left(4x^3 + \frac{1}{16x^3}\right)^2} \, dx
\]

\[
= \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right) \, dx
\]

\[
= \left(\frac{x^4}{4} + \frac{1}{32x^2}\right)_1^2
\]

\[
= 15 + \frac{3}{128}
\]