Limit of a Function

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The Legacy of Galileo, Newton, and Leibniz

• Galileo

  – was interested in falling bodies.
  – forged a new scientific methodology - *observe nature, construct experiments to test what you observe, and construct theories that explain the observations.*
• Newton

  – was able, using his new tools of calculus, to explain why falling bodies behave in this way: an object, falling under the influence of gravity, will have constant acceleration of $9.8 \text{m/sec}^2$.

  – his laws of motion and of universal gravitation drew under one simple mathematical theory Newton’s laws of falling bodies, Kepler’s laws of planetary motion, the motion of a simple pendulum, and virtually every other instance of dynamic motion observed in the universe.
- Leibnitz
  - co-inventor of calculus, took a slightly different point of view but also studied rates of change in a general setting.
Newton’s Question

• How do we find the velocity of a moving object at time $t$?

• What in fact do we mean by velocity of the object at the instant of time $t$?? We know how to find the average velocity of an object during a time interval $[t_1, t_2]$?
The average velocity during a time interval is the distance traveled divided by the elapsed time, i.e.

$$\text{Average velocity over}[t_1, t_2] = \frac{\text{distance traveled}}{t_2 - t_1}.$$
Definition

Let $x(t)$ be a function that gives the position at time $t$ of an object moving on the $x$-axis. Then

$$\text{Ave vel}[t_1, t_2] = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$\text{Velocity}(t) = \lim_{h \to 0} \frac{x(t + h) - x(t)}{h}.$$
Limit of a Function – Definition

We say that a function \( f \) approaches the limit \( L \) as \( x \) approaches \( a \), written \( \lim_{x \to a} f(x) = L \), if we can make \( f(x) \) as close to \( L \) as we please by taking \( x \) sufficiently close to \( a \).
Example
Theorem. The limit of $f$ as $x \to a$ exists if and only if both the right-hand and left-hand limits exist and have the same value. I.e.,

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$
Examples

Compute the limits:

- \( \lim_{x \to 2} \frac{x-2}{x+3} \)
- \( \lim_{x \to -1} \frac{x^2-1}{x-1} \)
- \( \lim_{x \to 0} \frac{1}{x} \)
**Theorem.** If \( \lim_{x \to a} f(x) = A \) and \( \lim_{x \to a} g(x) = B \) both exist, then

1. \( \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = A + B \)
2. \( \lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = A - B \)
3. \( \lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B \)
4. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B} \) \((B \neq 0)\).
Examples

1. \(\lim_{x \to 1} \frac{x^2 - 2x + 3}{x^3 + 3x - 1}\)

2. \(\lim_{x \to 0} \frac{|x|}{x}\)

3. Let \(f(x) = 1/x\). Compute \(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\)
Limits at Infinity

\[ \lim_{x \to \infty} f(x) = L \] means that the value of \( f(x) \) approaches \( L \) as the value of \( x \) approaches \( +\infty \). This means that \( f(x) \) can be made as close to \( L \) as we please by taking the value of \( x \) sufficiently large. Similarly, \( \lim_{x \to -\infty} f(x) = L \) means that \( f(x) \) can be made as close to \( L \) as we please by taking the value of \( x \) sufficiently small (in the negative direction).
Example

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$
More Examples

Evaluate the limits:

1. \( \lim_{x \to \infty} \frac{x - 1}{x^3 + 2} \)

2. \( \lim_{x \to \infty} \frac{3x^2 - 2x + 1}{4x^2 - 1} \)

3. \( \lim_{x \to \infty} \frac{x^4 - x^2 + 2}{x^3 + 3} \)
For the limit \( \lim_{x \to \infty} \frac{P(x)}{Q(x)} \), where \( P(x) \) is a polynomial of degree \( n \) and \( Q(x) \) is a polynomial of degree \( m \),

1. If \( n < m \), the limit is 0,

2. If \( n > m \), the limit is \( \pm \infty \),

3. If \( n = m \), the limit is the quotient of the coefficients of the highest powers.
Example

Evaluate the limit:

$$\lim_{x \to \infty} \frac{x}{\sqrt{3x^2 + 2}}$$
Compute the limit $\lim_{x \to 0} \frac{1}{x^2}$. 

\begin{figure}
\centering
\includegraphics{image.png}
\caption{Graph of $\frac{1}{x^2}$}
\end{figure}
Evaluate $\lim_{x \to \pi/2} \tan x$. 