The Aizenman-Kesten-Newman Theorem

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Abstract

The uniqueness of the infinite open cluster in the setting of bond percolation on the square grid was proven by Harris in 1960 [6]. As shown by Fisher in 1961 [4], Harris’ proof can be extended to include site percolation on the square grid. Aizenman, Kesten, and Newman [1] show that this fact is true in a much more general setting, as well.

Let $\Lambda$ be a connected, infinite, locally-finite, vertex-transitive graph, and take $\Omega = \{0, 1\}^{V(\Lambda)}$ to be the probability space. The Aizenman-Kesten-Newman theorem states that under these conditions, there can be at most one infinite open cluster. We will discuss the proof of this result discovered by Burton and Kean [2].

1 Introduction

The Aizenman-Kesten-Newman theorem, particularly when combined with Menshikov’s theorem, is an exceptional tool for simplifying a great variety of proofs. For instance, it allows for a simpler proof of the Harris-Kesten theorem, i.e. $p_H = p_T = 1/2$. [3]. It can also be used to prove that for any planar lattice $\Lambda$ (satisfying some symmetry conditions), $p^b_c(\Lambda) + p^b_c(\Lambda^*) = 1$, where $\Lambda^*$ is $\Lambda$’s dual lattice. There are many more such examples (see chapter 5 of Bollobás and Riordan [3]), but to see one example of how this theorem may be applied, let us consider the proof of the fact that for the triangular lattice $T$, $p^b_c(T) = 1/2$.

Suppose for contradiction that $p^b_c(T) < 1/2$ and consider a lattice with percolation probability $p = 1/2$. Let $H_n$ be the origin-centered hexagon with $n$ sites on each of the six sides, and number the sides of $H_n$ cyclically. Define the event $L_i$ to be “an infinite open path leaves from side $i$.” Similarly, let $L^*_i$ = “an infinite closed path leaves from side $i$,” and let $E = L_1 \cap L^*_2 \cap L_4 \cap L^*_5$. By Harris’ lemma [6], $\mathbb{P}(E) > 0$, and since $E$ is independent of the

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event $H_{n-1}$ (since the infinite paths only meet $H_n$ at its boundary), we have that $\mathbb{P}(E$ occurs and $H_{n-1}$ has all closed sites) $> 0$. Let $P_1, P_2^*, P_4$, and $P_5^*$ be the corresponding paths, as in Figure 1.

![Figure 1: $H_n$ with the paths $P_1, P_4, P_2^*$, and $P_5^*$.

Then $P_1$ and $P_4$ are disconnected infinite paths, contradicting the Aizenman-Kesten-Newman theorem. Thus $p^*_c(T) = p^*_H(T) = p^*_T(T) = 1/2$.

Define $I_k$ to be the event that there are exactly $k$ infinite open clusters (note that, a priori, $k$ may be infinite). To prove the uniqueness of the infinite cluster, we will need to use the result that for all $k$, $\mathbb{P}(I_k) \in \{0, 1\}$. Then we shall go through the proof that $\mathbb{P}(I_k) = 0$ for all $k \geq 2$, giving us our desired result.

2 Preliminary results

As mentioned, we need to use the result that $\mathbb{P}(I_k) \in \{0, 1\}$ for all $k$. This is true for all automorphism invariant events – that is, all events that are mapped into themselves by any automorphism induced by an automorphism of $\Lambda$ [3]. To see that $I_k$ is automorphism invariant, note that for any automorphism $\phi : \Lambda \to \Lambda$, two vertices $x$ and $y$ in $\Lambda$ are adjacent if and only if $\phi(x)$ and $\phi(y)$ are adjacent in $\phi(\Lambda)$. This will give us the following result of Newman and Schulman [7].

**Lemma 2.1.** For all $k \in [2, \infty)$, $\mathbb{P}(I_k) = 0$.  

Proof. Fix some vertex $x_0 \in V(\Lambda)$ and let $k \in \mathbb{N}$. Suppose that $\mathbb{P}(I_k) > 0$ (and therefore that $I_k$ holds). Define $T_{n,k}$ to be the event that $I_k$ holds and each infinite cluster contains a site in $B_n(x_0)$, the ball of radius $n$ around $x_0$. Since these balls cover $\Lambda$, we have that for large enough $n$, changing each closed site in $B_n(x_0)$ to open will connect the $k$ infinite clusters that meet $B_n(x_0)$. Therefore, $\mathbb{P}(I_1) > 0$. But then we have that $\mathbb{P}(I_k) = \mathbb{P}(I_1) = 1$, which tells us that $k = 1$. \qed

Before we continue to Burton and Keane’s theorem, we will need the following technical lemma about graphs [3].

**Lemma 2.2.** Let $G$ be a connected finite graph. Let $L = \{l_1, \ldots, l_t\}$, $C = \{c_1, \ldots, c_s\} \subseteq V(G)$ be disjoint, and suppose that deleting $c_i$ disconnects $G$ into components, at least 3 of which contain vertices of $L$. Then $t \geq 2 + s$.

**Proof.** Note that the “worst case” is when $G$ has the minimal amount of edges necessary to make it connected and to have $L, C \subseteq V(G)$. Hence, the case to consider is when $G$ is a tree whose leaves are contained in $L$. Then for all $i$, $d(c_i) \geq 3$. But we know that a tree has $2 + \sum_{v \in I(V)}(d(v) - 2)$ leaves, where $I(V)$ is the set of internal vertices, so

$$|L| \geq \text{(number of leaves)} \geq 2 + \sum_{i=1}^s (3 - 2) = 2 + s.$$ \qed

Note also that this result can be extended to disconnected graphs by considering one component at a time.

3 The uniqueness of the infinite open cluster

Define a graph $\Lambda$ to be amenable if, for all $x \in V(\Lambda)$,

$$\lim_{n \to \infty} \frac{|S_n(x)|}{|B_n(x)|} = 0.$$ 

Now we are ready to tackle Burton and Keane’s proof of the fact that there can be at most one infinite cluster.

**Theorem 3.1.** For any infinite, connected, locally finite, amenable, vertex-transitive graph $\Lambda$, $\mathbb{P}(I_k) = 0$ for all $k \geq 3$. 

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Proof. Suppose, for contradiction, that $\mathbb{P}(I_k) > 0$ for some $k \geq 3$. As before, fix a vertex $x_0 \in V(\Lambda)$. Fix $r$ such that $B_r(x_0)$ has a positive probability of containing sites from at least 3 infinite clusters. For $x \in V(\Lambda)$, let $T_r(x) =$ “all sites in $B_r(x)$ are open and there exists an infinite cluster $C$ such that when all sites in $B_r(x)$ are changed to closed, $C$ is disconnected into at least 3 infinite clusters.” Note that, by transitivity, for all $x \in V(\Lambda)$, $\mathbb{P}(T_r(x)) = a > 0$.

Let $W \subseteq B_{n-r}(x_0)$ be maximal subject to the constraint that the balls $B_{2r}(w)$ are disjoint for all $w \in W$. By the maximality of $W$, for any $z \in B_{n-r}(x_0) \setminus W$, $d(z, w) \leq 4r$ for some $w \in W$. Note that the balls $\{B_{4r}(w) \mid w \in W\}$ cover $B_{n-r}(x_0)$. Therefore

$$|W| \geq \frac{B_{n-r}(x_0)}{B_{4r}(x_0)}.$$

Using the fact that we can find a positive constant $c$ such that $|W| \geq c|B_{n+1}(x_0)|$ for all $n \geq r$, and that $\Lambda$ is amenable, we have $|W| \geq \frac{1}{a}|S_{n+1}(x_0)|$ for sufficiently large $n$. Let us fix such an $n$.

Define $B_r(w)$ to be a cut-ball if $w \in W \subseteq B_{n-r}(x_0)$ and $T_r(w)$ holds, i.e. $B_r(w) \subseteq B_n(x_0)$ and every site in $B_r(w)$ is open. Let $s$ be the number of cut-balls. Then

$$\mathbb{E}(s) = \sum_{w \in W} \mathbb{P}(T_r(w)) = a|W| \geq |S_{n+1}(x_0)|.$$

Therefore, $\mathbb{P}(s \geq |S_{n+1}(x_0)|) > 0$. For the remainder of this proof, consider a configuration $\omega$ under which $s \geq |S_{n+1}(x_0)|$.

Let $K$ be the union of all infinite clusters that meet $B_n(x_0)$. Change all of the sites in cut-balls to closed. Then $K$ is disconnected into infinite clusters $L_1, \ldots, L_t$ and finite clusters $F_1, \ldots, F_u$. Note $t \leq |S_{n+1}(x_0)|$ since each infinite cluster contains a site in the sphere $S_{n+1}(x_0)$. Let $C_1, \ldots, C_s$ be the cut-balls.

Now recall Lemma 2.2, where the graph $G$ is defined by contracting each $C_i, F_i$, and $L_i$ to a single vertex $c_i, f_i$, and $l_i$, respectively (as in Figure 2). The infinite components of $K$ correspond to components of $G$ containing at least one vertex in $L = \{l_1, \ldots, l_t\}$. Note that since $C_i$ is a cut-ball, deleting $c_i$ from $G$ disconnects a component into at least 3 components containing vertices of $L$. Applying Lemma 2.2 then says that

$$|S_{n+1}(x_0)| \geq t \geq s + 2 \geq |S_{n+1}(x_0)| + 2,$$

which is a contradiction. \qed
4 A possible direction for future work

One easy verification of the Aizenman-Kesten-Newman theorem for the special case of bond percolation in $\mathbb{Z}^2$ can be constructed as follows.

Suppose, for contradiction, that there are at least two infinite open clusters, $C_1$ and $C_2$. Choose $r \in \mathbb{N}$ such that $B_r(0)$, the square of radius $r$ centered at $(0,0)$, is the smallest (origin-centered) square that meets both $C_1$ and $C_2$. Now consider an annulus surrounding $B_r(0)$. By Harris’ lemma [6], each of the four rectangles comprising the annulus is crossed length-wise by an open path with some probability $\varepsilon > 0$ as in Figure 3.

![Figure 3: An annulus with open path P. [3]](image)

If such a path $P$ exists in each of the four rectangles, then $C_1$ and $C_2$ must be connected by an open path. Iteratively taking (proportional) annuli around the resulting square, we have (with probability 1) some annulus with a path connecting the two clusters.

Clearly this construction does not immediately generalize. However, perhaps one can find an extension of this proof to (for example) a three-dimensional hypercubic lattice, where the clusters would become surfaces.
References


