Ergodicity for Non-reversible Chains

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Abstract

A Markov chain is ergodic if for some positive integer $N$, it is possible to get from any state $i$ to any state $j$ in exactly $N$ steps. In this paper we establish an upper bound for the lowest value of such an $N$ by showing that for an ergodic $M$ state Markov chain we may go from any state $i$ to any state $j$ in $m$ steps, where $m \geq (M - 1)^2 + 1$. We then show how to construct a Markov chain of $M$ states such that this bound is met with equality.

1 Preliminaries

In this paper we consider Markov chains with at least two states, and we think of a given Markov chain as an unweighted directed graph. This is because we are only concerned about whether the probability of transition between a state $i$ and a state $j$ is greater than zero. In the undirected graph representation of the Markov chain, then, the nodes are the states of the Markov chain, and there is an edge going from node $i$ to node $j$ if the transition probability of going from state $i$ to state $j$ is non-zero. Note that from this point forward all variables are non-negative integers unless stated otherwise.

It is also important to note that if in a given Markov chain, we can go from any node $i$ to any node $j$ in exactly $N$ steps, then we can do the same in exactly $N + 1$ steps. To see this, take some arbitrary nodes $a$ and $b$ in the Markov chain and assume we can go from any node $i$ to any node $j$ in exactly $N$ steps. There must be some node $k$ in the Markov chain such that $b$ is a neighbour of $k$, since otherwise we could not get to $b$ from any state other than $b$, which would contradict our assumption. By our assumption, then, we have a walk $w$ from $a$ to $k$ of length $N$. We can therefore take a walk of length $N + 1$ from $a$ to $b$ by going from $a$ to $k$ along $w$ and then stepping from $k$ to $b$. We therefore have the above statement.

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From the preceding proof we then have that if we can get from a given node \( i \) to any node \( j \) in \( N \) steps, then we can do the same in exactly \( N + 1 \) steps, and it then follows from repeated application of this fact that we can get from \( i \) to \( j \) in exactly \( n \) steps for \( n \geq N \). Moreover, we get from repeated application of the statement from the previous paragraph that if we can go from any node \( i \) to any node \( j \) in exactly \( N' \) steps, then we can do the same in exactly \( n' \) steps for \( n' \geq N \).

2 The Theorem

**Theorem 1.** For an ergodic Markov chain with \( M \) states we may go from any state \( i \) to any state \( j \) in exactly \( m \) steps, for all \( m \geq (M - 1)^2 \).

**Proof.** Take some arbitrary ergodic Markov chain with \( M \) states, where \( M \geq 2 \), denoting it as \( G \). We may then choose a positive integer, denoting it \( N_G \), such that it is possible to get from any state \( i \) to any state \( j \) in exactly \( N \) steps in \( G \). We first show:

**Lemma 1.** Every node in \( G \) is on a cycle. \( G \) contains a cycle with \( \tau < M \) nodes.

**Proof.** Take some node \( i \in G \). Then we have that there is a walk from \( i \) back to \( i \) of length \( N_G \). Then if the only two places in the walk with the same node are the beginning and the end (since we start and end at \( i \)), we have a cycle. If on the other hand the walk is not a cycle, then there are two places in the walk, at least one of which is not the beginning or end, where we have the same node, call it \( y \). We may then replace these two instances of \( y \) and all nodes between them in the walk with the single node \( y \). Since the walk has finite length, we may repeat this process until we have a cycle. So we get that each \( i \in G \) is on a cycle.

Now suppose for the purpose of contradiction that none of the nodes in \( G \) are on a cycle of length \( \tau < M \). Then they must all be only on a cycle of length \( M \). \( G \), then, is a cycle of length \( M \) since introducing any edge to a cycle of length \( M \) would create a cycle of length less than \( M \). But a cycle of length greater than one is not ergodic. To see that this is true, take some \( i \in G \), and take some walk on the cycle starting at node \( i \). The walk is always at \( i \) after \( 0, M, 2M, \ldots \) steps and never at \( i \) at any other time. This is because at each step we only have one choice as to the next node we visit, so our walk is a walk around the cycle. So there cannot be an \( N \) such that we can get from \( i \) to \( i \) in exactly \( N \) steps and get from \( i \) to some node \( j \in G \) in exactly \( N \) steps, where \( i \neq j \). We therefore get that \( G \) is not ergodic, a contradiction. We can conclude, then, that \( G \) contains a cycle with \( \tau < M \) nodes. \( \square \)
Now take a node $a \in G$ that is on a cycle of length $\tau < M$ (we may do so by Lemma 1). Furthermore, let $T(m)$ denote the set of nodes accessible from $a$ in exactly $m$ steps. We then get that $T(m) \subseteq T(m + \tau)$. To see this suppose $j \in T(m)$. We then have that there is a walk of length $m$ from $a$ to $j$, which we denote as $w$. We can then get from $a$ to $j$ in exactly $m + \tau$ steps by first walking from $a$ to $a$ by going around the cycle (this takes $\tau$ steps) and then walking from $a$ to $j$ along $w$. So we get that $j \in T(m + \tau)$.

It then follows from repeated application of the above fact that $T(0) \subseteq T(0 + \tau) \subseteq T(0 + \tau + \tau) \subseteq ...$ where $T(0) = \{a\}$ (since $a$ is the only node accessible in 0 steps). The expression then simplifies to $T(0) \subseteq T(\tau) \subseteq T(2\tau) \subseteq ...$. Considering this series of inclusions, we now show that if $T(x\tau) = T((x + 1)\tau)$ for some positive integer $x$, then $T((x + 1)\tau) = T((x + 2)\tau)$. Assume that $T(x\tau) = T((x + 1)\tau)$ for some positive integer $x$. We have $T((x + 1)\tau) \subseteq T((x + 2)\tau)$ from our series of inclusions. Now to show that $T((x + 2)\tau) \subseteq T((x + 1)\tau)$, take some $i \in T((x + 2)\tau)$. We then have a walk of length $(x + 2)\tau$ in $G$ from $a$ to $i$, which we denote as $w$. Now consider the $((x + 1)\tau)$st node of $w$, which we denote as $k$. We get that $k \in T((x + 1)\tau)$ since we can reach it from $a$ in $(x + 1)\tau$ steps by following $w$. Also note that since $k$ is the $((x + 1)\tau)$st and $i$ is the $((x + 2)\tau)$st node on $w$, we can go from $k$ to $i$ in $(x + 1)\tau - (x + 2)\tau = \tau$ steps. Then take such a walk of length $\tau$ from $k$ to $i$ and denote it as $w'$. Since $T(x\tau) = T((x + 1)\tau)$, we get that $k \in T(x\tau)$. We therefore have a walk of length $x\tau$ from $a$ to $k$, which we denote as $w''$. We can then get from $a$ to $i$ in $(x + 1)\tau$ steps by first going from $a$ to $k$ in $x\tau$ steps along $w''$ and then going from $k$ to $i$ in $\tau$ steps by taking $w'$. We therefore have that $T((x + 2)\tau) \subseteq T((x + 1)\tau)$ so $T((x + 2)\tau) = T((x + 1)\tau)$.

Then by the above fact if $T(x\tau) = T((x + 1)\tau)$, then $T((x + 1)\tau) = T((x + 2)\tau)$, which then implies (again by the same fact) that $T((x + 2)\tau) = T((x + 3)\tau)$, which in turn implies $T((x + 3)\tau) = T((x + 4)\tau)$, and so on. So we get that if $T(x\tau) = T((x + 1)\tau)$ then $T(n\tau) = T((n + 1)\tau)$ for all $n \geq x$.

Now since $G$ has $M$ nodes and $T(0)$ has one node, we can have at most $M - 1$ strict containments in $T(0) \subseteq T(\tau) \subseteq T(2\tau) \subseteq ...$. So the first equality in the sequence can happen no later than $T((M - 1)\tau) \subseteq T((M)\tau)$. Since we have an equality in the sequence before or at $T((M - 1)\tau) \subseteq T((M)\tau)$, by the conclusion in the previous paragraph we get $T((M - 1)\tau) = T((M)\tau)$, and that $T(n\tau) = T((n + 1)\tau)$ for all $n \geq (M - 1)$. So by the transitivity of equality, $T(n\tau) = T((M - 1)\tau)$ for all $n \geq (M - 1)$.

Recalling the discussion in the Preliminaries, we have that for all $p \geq N_G$ we may go from any node $i$ to any node $j$ in exactly $p$ steps in $G$. We may therefore take $p'$ such that $p' \geq N_G$, $p' \geq (M - 1)\tau$ and $p'$ is divisible by $\tau$. We then get $\frac{p'}{\tau}$ is an integer. Because we may go from $a$ to any node $j \in G$ in exactly $p'$ steps, we get $T(p') = T((\frac{p'}{\tau})\tau)$ contains
all nodes of $G$. Then since $p' \geq (M - 1)\tau$, we get $\frac{p'}{\tau}$ is an integer larger than $M - 1$. So by the conclusion from the previous paragraph, we get $T(\frac{p'}{\tau}) = T((M - 1)\tau)$, which gives us that $T((M - 1)\tau)$ contains all nodes of $G$. Again recalling the discussion in the Preliminaries we get that for all $q \geq (M - 1)\tau$, $T(q)$ contains all nodes of $G$.

Given the way we chose $a$, we can therefore conclude that we may go from any node that is on a cycle of length $\tau < M$ to any other node in $G$ in $q$ steps, where $q \geq (M - 1)\tau$. So since $\tau \leq M - 1$, we get that if all nodes in $G$ are on a cycle of length $\tau < M$, then we may get from any node $i$ to any other node $j$ in $r$ steps, where $r \geq (M - 1)(M - 1)$.

Suppose, however, that there are $z$ nodes, where $z \geq 1$, which are not on a cycle of length $\tau < M$. By Lemma 1, since each node is on a cycle, these must only be on a cycle of length $M$. Then the cycles with less than $M$ nodes have at most $M - z$ nodes. Note that by Lemma 1 there is at least one node on a cycle of length $\tau < M$ since $G$ has a cycle of length $\tau < M$. Then because we may go from such a node, to any other node in $G$ in $q$ steps, where $q \geq (M - 1)\tau$ (we concluded this in the previous paragraph), we may go from such a node, to any other node in $G$ in $q'$ steps, where $q' \geq (M - 1)(M - z)$.

Now consider a node $b$ that is only on a cycle of length $M$. If we start at $b$ and walk on the cycle of length $M$ we will get to a node that is on a cycle of length $\tau < M$ (we know from before that such a node exists in $G$ and the cycle of length $M$ contains every node of $G$). And since there are $z$ nodes that are only on a cycle of length $M$, it will take us at most $z$ steps to get from $b$ to a node, call it $c$, that is on a cycle of length $\tau < M$. From $c$ we can then get to every other node in $q'$ steps, where $q' \geq (M - 1)(M - z)$. So for $b$ we can get to any other node in $(M - 1)(M - z) + z = M^2 - (z + 1)M + 2z$ or less steps. Recalling the discussion in the Preliminaries, we therefore have that we can get from $b$ to any other node in $q''$ steps, where $q'' \geq (M - 1)(M - z) + z$. Since $M \geq 2$, $M^2 - (z + 1)M + 2z$ is maximized when $z$ is smallest, namely when $z = 1$. We therefore have that $(M - 1)(M - z) \leq (M - 1)(M - z) + z \leq (M - 1)(M - 1) + 1$ (note $(M - 1)(M - z) \leq (M - 1)(M - 1) + 1$ for the case where the node is on a cycle of length $\tau < M$). So in the case where we have nodes in $G$ that are only on a cycle of length $M$, we get that we may go from any node $i$ to any other node $j$ in $r'$ steps, where $r' \geq (M - 1)(M - 1) + 1$ steps.

So then in all cases we have that we may get from any node $i$ to any other node $j$ in $m$ steps, where $m \geq (M - 1)(M - 1) + 1$ steps. Hence we have the proof.

\[\square\]
Consider the graph of $M$ nodes shown above. We first show that it is ergodic by showing that it is possible to get from any state $i$ to any state $j$ in exactly $M(M-1)$ steps. To do this first note that for any node $i$ in graph other than 1 we may go from $i$ to $i$ in exactly $M$ or $M-1$ steps (we call this going around the $M$ or $M-1$ cycle), depending on whether we take the edge $(M,2)$ or not on our walk. Now take some nodes $j,k$ in the graph. If $j$ and $k$ are the same node, we may get from $j$ to $k$ in exactly $(M-1)$ steps by walking around the cycle of length $M$, $M-1$ times. If $j$ and $k$ are not the same node then we have two cases. If $k$ is not 1 then we can get from $j$ to $k$ in $n$ steps by walking along the cycle of length $M$, where $n < M$. We then walk $n$ times around the cycle of length $M-1$, and $M-1-n$ times around the cycle of length $M$, so by the end of the walk we end up at $k$. This walk takes $n + n(M-1) + (M-1-n)(M) = n(M) + (M-1-n)(M) = (M-1)(M)$ steps. If, on the other hand $k$ is 1, then we may walk along the cycle of length $M-1$ from $j$ to $M$ in $n'$ steps where $n' < M-1$ (remember $j$ is not 1). Then walk $n'+1$ times around the cycle of length $M-1$, and $M-1-n'-1$ times around the cycle of length $M$, so we end up at $M$. Then step from $M$ to 1. This walk takes $n'+(n'+1)(M-1)+(M-1-n-1)(M)+1 = (n'+1)(M)+(M-1-n-1)(M) = (M-1)(M)$ steps. So we conclude that the graph is ergodic.

We now show that it is not possible to get from any state $i$ to any state $j$ in $(M-1)(M-1)$ steps, namely that we cannot get from 1 to 1 in $(M-1)(M-1)$ steps. Suppose for contradiction that this is possible. Consider a walk of length $(M-1)(M-1)$ starting from 1. For the first $M-1$ steps we must walk on the cycle of length $M$, until we reach node $M$. We may then walk as we please, but we must end up back at node $M$ since our final step must then be from $M$ to 1. So after getting to $M$ for the first time we walk around the cycle of length $M$ some $x$ number of times and around the cycle of length $M-1$ some $y$ number of times before getting to 1. We then get that $M-1 + xM + y(M-1) + 1 = (x+1)M + y(M-1) = (M-1)(M-1)$ from which it follows that $\frac{x+1}{M-1}M + y = M - 1$. Then since $gcd(M-1,M) = 1$, $0 < x + 1 < M - 1$ (if $x+1 \geq M-1$ we would have $(x+1)M > (M-1)^2$ which would be a contradiction), and $y$
is a non-negative integer, we get that $\frac{x+1}{M-1}M + y$ is not an integer. So $\frac{x+1}{M-1}M + y \neq M - 1$, a contradiction. We therefore see that our bound is met with equality.

References