On Bounding the Expected Progress of a Random Walk

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March 5, 2013

Abstract

We discuss the probabilistic proof (by Peyre) of a bound of Varopoulos and Carne on the probability of a random walk on a connected simple graph \( G \) starting at a vertex \( x \) and being at a vertex \( y \) at time \( t \). We also discuss some corollaries of this bound that enable us to find bounds on the probability that a random walk travels at least a certain distance in a given amount of time, and consequently the expected distance travelled by a random walk in \( t \) steps.

1 Introduction

The original bound was presented by Varopoulos in 1985 [5] and independently improved upon by Carne [1] in the same year. In both cases, spectral methods were used to prove the bound. Peyre [4], in 2007, found a probabilistic proof of this result. In all three cases, the bounds apply in the case of any countable point set \( V \) and any irreducible, reversible Markov chain \( X \) on \( V \) with transition matrix \( P = (p(x,y))_{x,y \in V} \). We present a version specific to random walks on a graph, but the difference is largely semantic, and it is not difficult to check that all of the arguments hold in the more general case. Furthermore, when we discuss graph distance from a given \( x \in V(G) \), i.e. the function \( d(x,\cdot) \), one may, if so inclined, consider instead any Lipschitz function \( \xi(\cdot) \) which satisfies \( \xi(x) = 0 \) and \( \xi(y) = \inf_{t \in \mathbb{N}} (p^t(x,y) \neq 0) \) for all \( y \in V \). Peyre also discusses distances more “flexible” than that of graph distance, and for more on that we invite the adventurous reader to read directly from that work.

2 Motivation

A statement of the Varopoulos-Carne bound in terms of a random walk on a graph is as follows.

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Theorem 2.1. Consider a connected, undirected, simple graph $G = (V, E)$. Let $P = (p(x, y))_{x,y \in V(G)}$ be the transition matrix associated with a simple random walk on $G$. Then

$$p^t(x, y) \leq \sqrt{e} \sqrt{\frac{\deg(y)}{\deg(x)}} \exp \left( -\frac{(d - 1)^2}{2(t - 1)} \right)$$

where $p^t(x, y)$ is the probability that a random walk beginning at $x$ is at vertex $y$ at time $t$, and $d = d(x, y)$ is the graph distance between $x$ and $y$.

Before we go on to discuss the proof of Theorem 2.1, we may want to take a moment to consider some uses of this bound. Suppose that we have a random walk $\{x_0, x_1, \ldots\}$ on a $G$ and we wish to find bounds on its progress by time $t$. The following bounds are direct corollaries of Theorem 2.1.

Corollary 2.2. For any (simple, undirected, connected) graph $G$ on $n$ vertices, any $t \in \mathbb{N}$, $c \in \mathbb{R}$, and any random walk $\{x_0, x_1, \ldots\}$ on $G$,

$$\mathbb{P}(d(x_0, x_t) > c\sqrt{t}) < n^{3/2} \exp \left( \frac{1}{2} - \frac{c^2 t}{2(t - 1)} \right).$$

Proof. By Theorem 2.1 and using the fact that there are fewer than $n$ vertices at any given distance from $x_0$,

$$\mathbb{P}(d(x_0, x_t) > c\sqrt{t}) = \sum_{y \in V(G): d(x_0, y) > c\sqrt{t}} p^t(x_0, y)$$

$$\leq \sum_{y \in V(G): d(x_0, y) > c\sqrt{t}} \sqrt{e} \sqrt{\frac{\deg(y)}{\deg(x)}} \exp \left( -\frac{(d(x, y) - 1)^2}{2(t - 1)} \right)$$

$$\leq \sum_{y \in V(G): d(x_0, y) > c\sqrt{t}} \sqrt{e} \sqrt{n} \exp \left( -\frac{c^2 t}{2(t - 1)} \right)$$

$$< \sqrt{e} n^{3/2} \exp \left( -\frac{c^2 t}{2(t - 1)} \right)$$

as desired. \qed

Corollary 2.3. For any (simple, undirected, connected) graph $G$ on $n$ vertices, any $t \in \mathbb{N}$, and any random walk $\{x_0, x_1, \ldots\}$ on $G$,

$$\mathbb{E}[d(x_0, x_t)] < 1 + \sqrt{(t - 1)(1 - 2 \ln(n^{-5/2}))}$$
Proof. Note that for any \( k \in \mathbb{N} \), \( \mathbb{E}[d(x_0, x_t)] \leq pn + (1 - p)k \) where \( p = \mathbb{P}(d(x_0, x_t) > k) \). By Corollary 2.2, we know that for any \( k \in \mathbb{N} \), \( p < n^{3/2} \exp \left( \frac{1}{2} - \frac{k^2}{2(t-1)} \right) \) so we have that for any \( k \in \mathbb{N} \),

\[
\mathbb{E}[d(x_0, x_t)] \leq pn + (1 - p)k < n^{5/2} \exp \left( \frac{1}{2} - \frac{k^2}{2(t-1)} \right) + k
\]

Then letting \( k = \sqrt{t - 1 - 2(t-1) \ln (n^{-5/2})} \), we have

\[
\mathbb{E}[d(x_0, x_t)] < n^{5/2} \exp \left( \frac{1}{2} - \frac{t - 1 - 2(t-1) \ln (n^{-5/2})}{2(t-1)} \right) + \sqrt{t - 1 - 2(t-1) \ln (n^{-5/2})}
\]

\[
= 1 + \sqrt{(t - 1)(1 - 2 \ln(n^{-5/2}))}
\]

\( \square \)

3 A probabilistic proof of the Varopoulos-Carne bound

**Lemma 3.1.** Given a filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{N}} \) and a real-valued process \( Y = (Y_t)_{t \in \mathbb{N}} \) consisting of increments of a martingale with respect to \( \mathcal{F} \) (i.e. \( Y_t = (Y_t)_{t \in \mathbb{N}} \) with \( \mathbb{E}[Y_t | \mathcal{F}_{t-1}] = 0 \)), let \( u \geq 0 \) be a fixed time and \( \lambda \in \mathbb{R} \) an arbitrary real number. Then

\[
\mathbb{E}[\exp(\lambda \sum_{t=1}^{u} Y_t)] \leq e^{u\lambda^2/2}
\]

**Proof.** We proceed by induction on \( u \). When \( u = 1 \) we have the statement \( \mathbb{E}[e^{\lambda Y_1}] \leq e^{\lambda^2/2} \). This follows directly from Hoeffding’s inequality (see [2]), which tells us that \( \mathbb{E}[e^{\lambda Y}] \leq e^{\lambda^2/2} \) for any random variable \( Y \) with \( \mathbb{E}[Y] = 0 \).

Now suppose that the result holds for \( u - 1 \) – that is, \( \mathbb{E}[\exp(\lambda \sum_{t=1}^{u-1} Y_t)] \leq e^{(u-1)\lambda^2/2} \). Then

\[
\mathbb{E}[\exp(\lambda \sum_{t=1}^{u} Y_t)] = \mathbb{E}[\exp(\lambda \sum_{t=1}^{u-1} Y_t) \mathbb{E}[e^{\lambda Y_u} | \mathcal{F}_{u-1}]]
\]

\[
\leq \mathbb{E}[\exp(\lambda \sum_{t=1}^{u-1} Y_t)] \mathbb{E}[e^{\lambda Y_u} | \mathcal{F}_{u-1}]] ||e^{\lambda Y_u} | \mathcal{F}_{u-1}||_{\infty} \text{ since } || \cdot ||_{\infty} \text{ is the sup norm}
\]

\[
\leq e^{(u-1)\lambda^2/2} ||\mathbb{E}[e^{\lambda Y_u} | \mathcal{F}_{u-1}]]||_{\infty} \text{ by the induction hypothesis}
\]

\[
\leq e^{(u-1)\lambda^2/2} e^{\lambda^2/2} \text{ by Hoeffding’s inequality}
\]

\[
= e^{u\lambda^2/2}
\]
as desired. \[\square\]

**Lemma 3.2.** Let \(X\) be any random variable and suppose that \(\mathbb{E}[e^{\lambda X}] \leq e^{\lambda^2/2}\) for any \(\lambda \in \mathbb{R}\). If \(A\) is an event such that \(\mathbb{E}[X | A] \geq c\) for some \(c \in \mathbb{R}_{\geq 0}\) then \(p = \mathbb{P}(A) \leq e^{-c^2/2k}\).

**Proof.** Jensen’s inequality (see [3]) for concave functions tells us that \(\ln(\mathbb{E}[e^{\lambda X} | A]) \geq \mathbb{E}[\ln(e^{\lambda X}) | A] = \mathbb{E}[\lambda X | A]\). Therefore \(\mathbb{E}[X | A] \leq \frac{1}{\lambda} \ln(\mathbb{E}[e^{\lambda X} | A])\). By assumption, we have that \(\mathbb{E}[e^{\lambda X} | A] \leq \frac{1}{p} e^{k\lambda^2/2}\). Therefore \(\mathbb{E}[X | A] \leq \frac{1}{\lambda} \ln(\frac{1}{p} e^{k\lambda^2/2})\). So then \(c \leq \ln(\frac{1}{p} e^{k\lambda^2/2})\) and therefore \(p \leq e^{-c\lambda + k\lambda^2/2}\). Letting \(\lambda = \frac{c}{k}\) yields the desired result. \[\square\]

Now we are ready to prove Theorem 2.1.

**Proof.** Note that \(p^t(x,y) = \frac{\deg(y)}{\deg(x)}\) by the reversibility of the simple random walk on an undirected graph. Therefore we can restate Theorem 2.1 to say \(\sqrt{p^t(x,y)p^t(y,x)} \leq \sqrt{e^{\exp\left(-\frac{(d-1)^2}{2(t-1)}\right)}}\).

Consequently, it suffices to show that we can bound \(p^t(x,y)p^t(y,x)\) by the slightly stronger \(e^{\exp\left(-\frac{(d-1)^2}{t-1}\right)}\).

Define the process \(M = (M_u)_{u \geq 1}\) by

\[
M_u = \sum_{s=1}^{u-1} (d(x,X_s) - \mathbb{E}_{X_s}[d(x,X_{s+1})])
\]

(1)

Note that each term in this sum is the expected change in distance from \(x\) from one step of the random walk to the next, so it would not be surprising if \(M\) behaved like a martingale. In fact,

\[
\mathbb{E}[M_u | \mathcal{F}_{u-1}] = \mathbb{E}[d(x,X_u) - \mathbb{E}_{X_{u-1}}[d(x,X_u)] - \sum_{s=1}^{u-2} (d(x,X_s) - \mathbb{E}_{X_s}[d(x,X_{s+1})]) | \mathcal{F}_{u-1}] = \mathbb{E}[d(x,X_u) | \mathcal{F}_{u-1}] - \mathbb{E}_{X_{u-1}}[d(x,X_u)|\mathcal{F}_{u-1}] + M_{u-1} = M_{u-1}
\]

and so \(M\) is indeed a martingale with respect to the filtration \(\mathcal{F}_u = \{X_1, X_2, \ldots, X_u\}\).

Let \(X^x\) and \(X^y\) be the versions of the chain \(X\) starting at \(X^x_0 = x\) and \(X^y_0 = y\) respectively, with associated processes \(M^x\) and \(M^y\) defined to be analogous to the definition in (1). Then we have
\[ \mathbb{E}_x[M^x_t|X^x_t = y] \geq d - 1 - \mathbb{E}_x \left[ \sum_{s=1}^{t-1} (\mathbb{E}X^x_s[d(x, X^x_{s+1})] - d(x, X^x_s)) \right] \] (2)

since \( d(x, X^x_t) = d \) and \( d(x, X^t) \) is 0 or 1.

Similarly,

\[ \mathbb{E}_y[M^y_t|X^y_t = x] \leq -(d - 1) - \mathbb{E}_y \left[ \sum_{s=1}^{t-1} (\mathbb{E}X^y_s[d(x, X^y_{s+1})] - d(x, X^y_s)) \right] \] (3)

For notational convenience, let us call the terms inside the sum of (2) \( m^x(s) \) and the terms inside the sum of (3) \( m^y(s) \), so that we can rewrite (2) and (3) as

\[ \mathbb{E}_x[M^x_t|X^x_t = y] \geq d - 1 - \mathbb{E}_x \left[ \sum_{s=1}^{t-1} m^x(s) \right] \] (4)

and

\[ \mathbb{E}_y[M^y_t|X^y_t = x] \leq -(d - 1) - \mathbb{E}_y \left[ \sum_{s=1}^{t-1} m^y(s) \right] \] (5)

respectively.

By reversibility, for all \( s \in [t-1] \) we have \( \mathbb{E}_x[m^x(s)|X^x_t = y] = \mathbb{E}_y[m^y(t-s)|X^y_t = x] \).

Therefore subtracting equation (5) from (4) yields

\[ \mathbb{E}_{x \otimes y}[M^x_t - M^y_t|X^x_t = y \text{ and } X^y_t = x] \geq 2(d - 1) \] (6)

Now we may apply Lemma 3.1 to the random variable \( M^x_t - M^y_t \) to get that \( \mathbb{E}_{x \otimes y}[e^{\lambda(M^x_t - M^y_t)}] \leq e^{(t-1)\lambda^2} \) (these details are straightforward and are left as an exercise to the reader). Finally, equation (6) allows us to apply Lemma 3.2 with \( k = 2(t - 1) \) to get

\[ p^x(x, y)p^y(y, x) = \mathbb{P}(X^x_t = y \text{ and } X^y_t = x) \leq \exp \left( -\frac{4(d - 1)^2}{2(t - 1)} \right) = \exp \left( -\frac{(d - 1)^2}{t - 1} \right). \]

\[ \square \]

References


[3] J. L. W. V. Jensen, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, 


225–252.