Interval-Vector Polytopes

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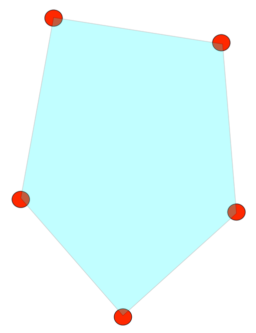
2012 Mathematical Sciences Research Institute - Undergraduate Program

Background



A **convex polytope** is formed by taking the **convex hull** of a set $A = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$, $\text{conv}(A)$, which is defined as

$$\left\{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \middle| \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} \lambda_i = 1\right\}.$$

A **simplex** is an n-dimensional polytope with $n+1$ vertices.

Given an n-dimensional polytope $\mathcal{P}$ with f_k k-dimensional faces, the **f-vector** of $\mathcal{P}$ is written as

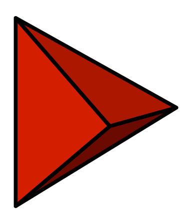
$$f(\mathcal{P}) := (f_0, f_1, \ldots, f_{n-1}).$$

Example 1. *Tetrahedron*

Simplex: 3-dimensional polytope, 4 vertices

f-vector: (4,6,4)

4 vertices, 6 edges, 4 planes



Denote the volume of a polytope $\mathcal{P}$ as $\text{vol}(\mathcal{P})$.

An **interval vector** [1] is a $(0,1)$-vector $x \in \mathbb{R}^n$ such that, if $x_i = x_k = 1$ for $i < k$, then $x_j = 1$ for every $i \leq j \leq k$.

Example 2. *Interval vectors*

$$(1,1,0),\ (0,0,0,0),\ (0,0,1,1,1,1,0)$$

Let $\alpha_{i,j} := e_i + e_{i+1} + \cdots + e_j$ for $i \leq j$ where e_i is the i^{th} standard unit vector.

Example 3. *For* $n = 6$

$$e_2 = (0,1,0,0,0,0)$$

$$e_5 = (0,0,0,0,1,0)$$

$$\alpha_{2,5} = (0,1,1,1,1,0)$$

Abstract: An interval vector is a $(0,1)$-vector where all the ones appear consecutively. Polytopes whose vertices are among these vectors have some astonishing properties. We present a number of interval-vector polytopes, including one class whose volumes are the Catalan numbers and another class whose volumes are the even numbers and face numbers mirror Pascal's triangle.

1. Complete Interval-Vector Polytope

Let $\mathcal{I}_n = \{\alpha_{i,j} | i,j \in [n], i \leq j\}$.

The **complete interval-vector polytope** is defined as $\mathcal{P}_{\mathcal{I}_n} := \text{conv}(\mathcal{I}_n)$.

We form a lattice-preserving bijection between the complete interval-vector polytope and Postnikov's complete root polytope in [2].

Theorem 1. *The volume of the n-dimensional interval-vector polytope is the nth Catalan number.*

$$\text{vol}(\mathcal{P}_{\mathcal{I}_n}) = \frac{1}{n+1}\binom{2n}{n}.$$

2. Fixed Interval-Vector Polytope

Given an interval length i and a dimension n we define the **fixed interval-vector polytope** $\mathcal{Q}_{n,i}$ as the convex hull of all vectors in $\mathbb{R}^n$ with interval length i

$$\mathcal{Q}_{n,i} := \text{conv}(\{\alpha_{j,j+i-1} | j \leq n-i+1\}).$$

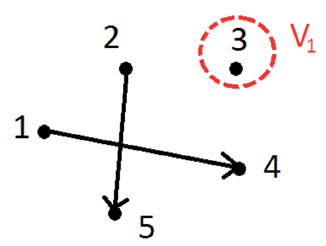
We project $\mathcal{Q}_{n,i}$ down to its ambient dimension and prove using Dahl's flow-dimension graph and the Cayley-Menger determinant

Theorem 2. *$\mathcal{Q}_{n,i}$ is an $(n-i)$-dimensional unimodular simplex.*

Example 4. *The fixed interval-vector polytope with $n = 5$, $i = 3$ is*

$$\mathcal{Q}_{5,3} = \text{conv}\left((1,1,1,0,0), (0,1,1,1,0), (0,0,1,1,1)\right).$$

Flow-dimension graph of $\mathcal{Q}_{5,3}$:



3. Interval Pyramid

Given a dimension n, define $\mathcal{P}_{n,1}$ to be the convex hull of all vectors in $\mathbb{R}^n$ with interval length 1 or $n-1$.

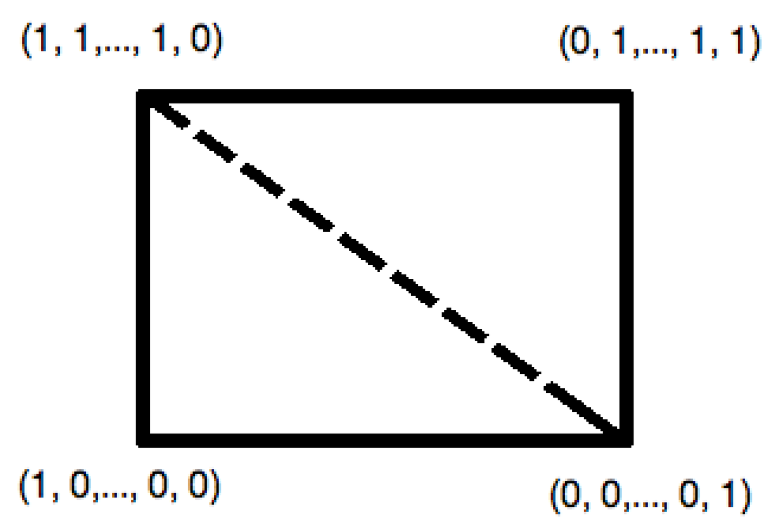
Example 5. *For* $n = 4$,

$$\mathcal{P}_{4,1} = \text{conv}\left((1,0,0,0), (0,1,0,0), (0,0,1,0),\right.$$
$$\left.(0,0,0,1), (1,1,1,0), (0,1,1,1)\right).$$

Theorem 3. *The f-vector for $\mathcal{P}_{n,1}$ for $n \geq 3$ is the n^{th} row of the Pascal 3-triangle.*

$n = 1$:	3					
$n = 2$:	4	4				
$n = 3$:	5	8	5			
$n = 4$:	6	13	13	6		
$n = 5$:	7	19	26	19	7	
$n = 6$:	8	26	45	45	26	8

Triangulation of the base of $\mathcal{P}_{n,1}$:



We apply the Cayley-Menger determinant to each simplex formed by pyramiding over the triangulation of $\mathcal{P}_{n,1}$ to prove:

Theorem 4. $\text{vol}\left(\mathcal{P}_{n,1}\right) = 2(n-2)$ *for* $n \geq 3$

4. Generalized Interval Pyramid

Due to the interesting properties of $\mathcal{P}_{n,1}$, we studied a related class of polytopes

$$\mathcal{P}_{n,i} := \text{conv}(e_1, \ldots, e_n, \alpha_{1,n-i}, \alpha_{2,n-i+1}, \ldots, \alpha_{i+1,n})$$

where $n > 2$ and $i \leq \frac{n}{2}$.

Example 6. *For $n = 5$ and $i = 2$*

$$\mathcal{P}_{4,2} = \text{conv}\left((1,0,0,0), (0,1,0,0), (0,0,1,0),\right.$$
$$\left.(0,0,0,1), (1,1,0,0), (0,1,1,0), (0,0,1,1)\right)$$

Proposition 1. *The dimension of $\mathcal{P}_{n,i}$ is n.*

Proposition 2. *Let*

$$\mathcal{B} = \text{conv}\left(\{e_1, e_2, \ldots, e_i, e_{n-i+1}, \ldots, e_n,\right.$$
$$\left.\alpha_{1,n-i}, \alpha_{2,n-i-1} \ldots, \alpha_{i+1,n}\}\right).$$

Then adding each vector in $\{e_{i+1}, e_{i+2}, \ldots, e_{n-i}\}$ sequentially pyramids over the previous base.

Finally, we have conjectured the volume of $\mathcal{P}_{n,i}$ and plan to prove it by proving a triangulation of the base of $\mathcal{P}_{n,i}$ contains 2^i simplices whose volume as one pyramids over them is $n - (i+1)$.

Conjecture 1.

$$\text{vol}(\mathcal{P}_{n,i}) = 2^i(n - (i+1)).$$

References

[1] Geir Dahl. Polytopes related to interval vectors and incidence matrices. *Linear Algebra Appl.*, 435(11):2955–2960, 2011.

[2] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.

Acknowledgement

This work was supported by NSF grant DMS-1156499 and NSA grant H98230-11-1-0213.