Thayer Prize Exam in Mathematics

for Dartmouth First-year students

Saturday May 30, 2020

PRINT NAME: ________________________________

Acknowledgment: Some of the problems are inspired by problems in recent math competitions in the US and in Russia, and by problems from other sources.

Honor Code: you are not allowed to give or receive any help on this exam. Using calculators or computers is not allowed. You have 3 hours to work on the exam and you can choose any 3 sequential hours to do this.

Exam Submission: When you stop working on the exam after 3 hours, you should scan your exam with handwritten solutions into one PDF file and email it back to

Vladimir.Chernov@dartmouth.edu

Your Exam should be received by the end of the day (Eastern Daylight Time) on which you took the exam.

Grader's use only

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Total: __________ /80

Problem 1. You are given a triangle that is **not** isosceles. Show that it is possible either to increase or to decrease the length of all the three sides of the triangle by the same number of units to get a triangle with a ninety degree angle.

Problem 2. Is $4^9 + 6^{10} + 3^{20}$ a prime number?

Problem 3. *F*ind all the numbers $x, y, z \in \left[0, \frac{\pi}{2}\right]$ that satisfy the system of equations $\sin x \cos y = \sin z$ and $\cos x \sin y = \cos z$.

Problem 4. A square is subdivided into n smaller rectangles whose sides are parallel to the sides of the square, in such a way that each line parallel to one of the sides of the square that does not contain a side of any of these smaller rectangles intersects exactly three of the rectangles. Is it possible to find such subdivision with $n = 8$? Prove your answer. For example such a subdivision into $n = 9$ rectangles is a 3×3 "chess" board.

Problem 5. For n any positive integer, show that $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) \leq n^n$.

Problem 6. A vector field $\mathbf{F}$ on a region $U \subseteq \mathbb{R}^3$ is *irrotational* if its curl is identically zero: $\operatorname{curl} \mathbf{F} = \mathbf{0}$. Let $f : U \to \mathbb{R}$ be a differentiable function whose derivative $Df(\mathbf{p})$ is never zero on U, and suppose that $\mathbf{F}$ is an irrotational vector field on U that is everywhere orthogonal to the level surfaces of f. Show that $f\mathbf{F}$ is also irrotational.

Problem 7. Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$f(x)^2 = \int_0^x (f(t)^2 + f'(t)^2)\, dt + 2020.$$

Problem 8. Evaluate $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2$.