How Many Borel Sets are There?

**Object.** This series of exercises is designed to lead to the conclusion that if $\mathcal{B}_\mathbb{R}$ is the $\sigma$-algebra of Borel sets in $\mathbb{R}$, then

$$\text{Card}(\mathcal{B}_\mathbb{R}) = \mathfrak{c} := \text{Card}(\mathbb{R}).$$

This is the conclusion of problem 4. As a bonus, we also get some insight into the “structure” of $\mathcal{B}_\mathbb{R}$ via problem 2. This just scratches the surface. If you still have an itch after all this, you want to talk to a set theorist. This treatment is based on the discussion surrounding [1, Proposition 1.23] and [2, Chap. V §10 #31].

For these problems, you will definitely want to have a close look at [1, §0.4] on well ordered sets. Note that by [1, Proposition 0.18], there is an uncountable well ordered set $\Omega$ such that for all $x \in \Omega$, $I_x := \{ y \in \Omega : y < x \}$ is countable. The elements of $\Omega$ are called countable ordinals. We let $1 := \inf \Omega$. If $x \in \Omega$, then $x + 1 := \inf \{ y \in \Omega : y > x \}$ is called the immediate successor of $x$. If there is a $z \in \Omega$ such that $z + 1 = x$, then $z$ is called the immediate predecessor of $x$. If $x$ has no immediate predecessor, then $x$ is called a limit ordinal.\(^1\)

1. Show that $\text{Card}(\Omega) \leq \mathfrak{c}$. (This follows from [1, Propositions 0.17 and 0.18]. Alternatively, you can use transfinite induction to construct an injective function $f : \Omega \to \mathbb{R}$.\(^2\)

   **ANS:** Actually, this follows almost immediately from Folland’s Proposition 0.17. By the Well Ordering Principle (Theorem 0.3 in Folland), we can assume that $\mathbb{R}$ is well ordered. Then, with this order, $\mathbb{R}$ cannot be isomorphic to an initial segment of $\Omega$ because $\mathbb{R}$ is uncountable and every initial segment in $\Omega$ is countable. Therefore $\Omega$ is either isomorphic to $\mathbb{R}$ or order isomorphic to an initial segment in $\mathbb{R}$. In either case, $\text{Card}(\Omega) \leq \text{Card}(\mathbb{R}) := \mathfrak{c}$.

2. If $X$ is a set, let $\mathcal{P}(X)$ be the set of subsets of $X$ — i.e., $\mathcal{P}(X)$ is the power set of $X$. Let $\mathcal{E} \subset \mathcal{P}(X)$. The object of this problem is to give a “concrete” description of the $\sigma$-algebra $\mathcal{M}(\mathcal{E})$ generated by $\mathcal{E}$. (Of course, we are aiming at describing the Borel sets in $\mathbb{R}$ which are generated by the collection $\mathcal{E}$ of open intervals.) For convenience, we assume that $\emptyset \in \mathcal{E}$.

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\(^1\)The set of countable ordinals has a rich structure. We let $2 := 1 + 1$, and so on. The set $\{ n \in \mathbb{N} \} \subset \Omega$ is countable, and so has a supremum $\omega$ (see [1, Proposition 0.19]). Then there are ordinals $\omega + 1, \omega + 2, \ldots, 2\omega, 2\omega + 1, \ldots, \omega^2, \omega^2 + 1, \ldots, \omega^\omega$, and so on.

\(^2\)The issue of whether or not $\text{Card}(\Omega) = \mathfrak{c}$ is the continuum hypothesis, and so is independent of the usual (ZFC) axioms of set theory.
Let
\[ \mathcal{E}^c := \{ E^c : E \in \mathcal{E} \} \quad \text{and} \quad \mathcal{E}_\sigma = \bigcup_{i=1}^\infty E_i : E_i \in \mathcal{E} \].

(Note, I just mean that \( \mathcal{E}_\sigma \) is the set of sets formed from countable unions of elements of \( \mathcal{E} \). Since \( \emptyset \in \mathcal{E}, \mathcal{E} \subset \mathcal{E}_\sigma \).

We let \( \mathcal{F}_1 := \mathcal{E} \cup \mathcal{E}^c \). If \( x \in \Omega \), and if \( x \) has an immediate predecessor \( y \), then we set
\[ \mathcal{F}_x := (\mathcal{F}_y)_\sigma \cup ((\mathcal{F}_y)_\sigma)^c. \]
If \( x \) is a limit ordinal, then we set
\[ \mathcal{F}_x := \bigcup_{y < x} \mathcal{F}_y. \]

We aim to show that
\[ \mathcal{M}(\mathcal{E}) = \bigcup_{x \in \Omega} \mathcal{F}_x \] \((\dagger)\)

(a) Observe that \( \mathcal{F}_1 \subset \mathcal{M}(\mathcal{E}) \).

(b) Show that if \( F_y \subset \mathcal{M}(\mathcal{E}) \) for all \( y < x \), then \( F_x \subset \mathcal{M}(\mathcal{E}) \). Then use transfinite induction to conclude that \( \mathcal{F}_x \subset \mathcal{M}(\mathcal{E}) \) for all \( x \in \Omega \).

(c) Show that the right-hand side of \((\dagger)\) is closed under countable unions.

(d) Conclude that \( \bigcup_{x \in \Omega} \mathcal{F}_x \) is a \( \sigma \)-algebra, and that \((\dagger)\) holds.

ANS: Since \( \mathcal{M}(\mathcal{E}) \) is a \( \sigma \)-algebra — and hence is closed under countable unions and complementation — it is clear that \( \mathcal{F}_1 \subset \mathcal{M}(\mathcal{E}) \). Thus if \( A = \{ x \in \Omega : \mathcal{F}_x \subset \mathcal{M}(\mathcal{E}) \} \), we certainly have \( 1 \in A \). Now suppose that \( y \in A \) for all \( y < x \). If \( x = z + 1 \), then because \( \mathcal{M}(\mathcal{E}) \) is a \( \sigma \)-algebra,
\[ F_x = (F_z)_\sigma \cup ((F_z)_\sigma)^c \subset \mathcal{M}(\mathcal{E}). \]

But if \( x \) is a limit ordinal, then trivially,
\[ F_x = \bigcup_{y < x} F_y \subset \mathcal{M}(\mathcal{E}). \]

Then it follows by transfinite induction (Folland, Proposition 0.15) that \( A = \Omega \). Therefore \( \bigcap_{x \in \Omega} F_x \subset \mathcal{M}(\mathcal{E}) \).
Now I claim that \( \bigcup_{x \in \Omega} F_x \) is a \( \sigma \)-algebra. Since it clearly contains \( \emptyset \) and is closed under complementation, it suffices to see that it is closed under countable unions. So, suppose that \( \{ E_j \}_{j=1}^{\infty} \subset \bigcup_{x \in \Omega} F_x \). Say, \( E_j \in F_{x_j} \). Since \( \{ x_j \} \) is countable, there is an \( x \in \Omega \) such that \( x_j \leq x \) for all \( j \) by Folland’s Proposition 0.19.\(^3\) Then, since \( \Omega \) has no largest element,

\[
\bigcup_{j=1}^{\infty} E_j \subset (F_x)_\sigma \subset F_{x+1} \subset \bigcup_{x \in \Omega} F_x.
\]

This shows that \( \bigcup_{x \in \Omega} F_x \) is a \( \sigma \)-algebra containing \( \mathcal{E} \). Hence

\[
\mathcal{M}(\mathcal{E}) \subset \bigcup_{x \in \Omega} F_x \subset \mathcal{M}(\mathcal{E}).
\]

Thus, (†) follows, and this completes the proof.

3. Recall that if \( A \) and \( B \) are sets, then \( \prod_{a \in A} B \) is simply the set of functions from \( A \) to \( B \). For reasons that are unclear to me, this set is usually written \( B^A \). Notice that \( \prod_{i=1}^{\infty} B = \prod_{i \in \mathbb{N}} B \) is just the collection of sequences in \( B \). Notice also that \( \text{Card}(B^A) \) depends only on \( \text{Card}(A) \) and \( \text{Card}(B) \).

(a) Check that

\[
\prod_{i=1}^{\infty} \left( \prod_{j=1}^{\infty} B \right) = \prod_{(i,j) \in \mathbb{N} \times \mathbb{N}} B.
\]

Thus the cardinality of either side of (*) is the same as \( \prod_{i=1}^{\infty} B \).

(b) Use these observations together with the fact that \( \text{Card}(\prod_{i=1}^{\infty} \{0, 1\}) = \mathfrak{c} := \text{Card}(\mathbb{R}) \) (which follows from [1, Proposition 0.12]) to show that

\[
\text{Card} \left( \prod_{i=1}^{\infty} \mathbb{R} \right) = \mathfrak{c}.
\]

(c) Show that if \( \text{Card}(\mathcal{E}) = \mathfrak{c} \), then \( \text{Card}(\mathcal{E}_\alpha) = \mathfrak{c} \).

\textbf{ANS:} The proof of (a) is immediate from the fact that \( \text{Card}(\mathbb{N} \times \mathbb{N}) = \text{Card}(\mathbb{N}) \). For (b), just note that

\[
\text{Card} \left( \prod_{j=1}^{\infty} \mathbb{R} \right) = \text{Card} \left( \prod_{j=1}^{\infty} \left( \prod_{i=1}^{\infty} \{0, 1\} \right) \right),
\]

which by part (a) has the same cardinality as \( \prod_{i=1}^{\infty} \{0, 1\} \). This proves (b).

\( ^3 \)This is the property of \( \Omega \) that is crucial here! Especially notice that \( \mathbb{N} \) does \textbf{not} have this property. This is why we need countable ordinals to describe \( \mathcal{M}(\mathcal{E}) \)
For (c), we have $E \subset E_\sigma$, so $\text{Card}(E) \leq \text{Card}(E_\sigma)$. But we have an obvious map of $\prod_{j=1}^\infty E$ onto $E_\sigma$. Thus $\text{Card}(E_\sigma) \leq \text{Card}(\prod_{j=1}^\infty E) = \text{Card}(\prod_{j=1}^\infty R)$, and the latter is bounded by $\mathfrak{c}$ in view of part (b). This completes the proof.

4. Let $\mathcal{B}_R$ be the $\sigma$-algebra of Borel sets in $R$. In [1, Proposition 0.14(b)], it is shown that if $\text{Card}(A) \leq \mathfrak{c}$ and if $\text{Card}(Y_x) \leq \mathfrak{c}$ for all $x \in A$, then $\bigcup_{x \in A} Y_x$ has cardinality bounded by $\mathfrak{c}$. By following the given steps, use this observation, as well as problems 2 and 3, to show that

$$\text{Card}(\mathcal{B}_R) = \mathfrak{c}.$$ (4)

(a) Let $\mathcal{E}$ be the collection of open intervals (including the empty set) in $R$. Then $\text{Card}(\mathcal{E}) = \mathfrak{c}$.

(b) $\mathcal{B}_R = \mathcal{M}(\mathcal{E})$.

(c) Define $\mathcal{F}_x$ as in problem 2. Use transfinite induction and problem 3 to prove that $\text{Card}(\mathcal{F}_x) = \mathfrak{c}$ for all $x \in \Omega$.

(d) Use problem 2 to conclude that $\mathcal{M}(\mathcal{E}) = \mathcal{B}_R$ has the cardinality claimed in (4).4

ANS: Parts (a) and (b) are immediate. For $\mathfrak{c}$, start by letting $A = \{ x \in \Omega : \text{Card}(\mathcal{F}_x) = \mathfrak{c} \}$. It follows from Problem 3(c), that $1 \in A$. Now suppose that $y \in A$ for all $y < x$. If $x = z + 1$, then $F_x \in A$ by Problem 3(c) again. If $x$ is a limit ordinal, then $x \in A$ by the observation the countable union of sets of cardinality $\mathfrak{c}$ has cardinality $\mathfrak{c}$. Thus $A = \Omega$ by transfinite induction.

Now problem 2 implies that $\mathcal{B}_R = \bigcup_{x \in \Omega} F_x$. Since each $F_x$ has cardinality $\mathfrak{c}$ and since $\Omega$ has cardinality at most $\mathfrak{c}$, the union has cardinality at most $\mathfrak{c}$ (Folland’s Proposition 0.14(b)). This completes the proof.

References


4It is my understanding that the classes $\mathcal{F}_x$ are all distinct; that is, $\mathcal{F}_x \subsetneq \mathcal{F}_y$ if $x < y$ in $\Omega$. But I don’t have a reference or a proof at hand.