Midterm for Math 103
Due Friday, November 14, 2008

Work on one side of $8\frac{1}{2} \times 11$ inch paper only. Start each problem on a separate page. (This last requirement can be waived for those \LaTeX users whose very short and elegant solutions would result in an uncomfortable waste of paper.)

1. Let $X$ be an uncountable set and let $\mathcal{M}$ be the connection of sets $E$ in $X$ such that either $E$ or $E^c$ is at most countable.

   (a) Show that $\mathcal{M}$ is a $\sigma$-algebra.

   (b) Show that

   $$\mu(E) := \begin{cases} 1 & \text{if } E \text{ is uncountable, and} \\ 0 & \text{otherwise} \end{cases}$$

   is a measure on $(X, \mathcal{M})$.

   (c) Describe the $\mathcal{M}$-measurable functions $f : X \to \mathbb{R}$ and their integrals.

2. Prove the “missing” results:

   (a) Lemma 69: If $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions which converges to a measurable function $f$ in measure, then every subsequence also converges to $f$ in measure.

   (b) Theorem 70: Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions which converges to a measurable function $f$ in measure and that $g \in \mathcal{L}^1(X)$ is such that, for each $n$, $|f_n(x)| \leq g(x)$ for almost all $x$. Then prove that $f_n \to f$ in $L^1(X)$.

   (Part (a) is really very straightforward. It is assigned as more of a hint for the second part than for any other reason.)

3. If $f_n \to f$ pointwise almost everywhere, then must $f_n \to f$ in measure? Does you conclusion change if “almost everywhere” convergence is replace by pointwise convergence everywhere? What if $\mu(X) < \infty$? (Assume that each of $f_n$ and $f$ are measurable.)

(a) Show that both the Monotone Convergence Theorem and Fatou’s Lemma are false without the assumption that the $f_n$ are nonnegative (at least almost everywhere).

(b) Show that Egoroff’s Theorem fails if we drop that assumption that $\mu(X) < \infty$.

5. Suppose that $\mu$ is $\sigma$-finite and that $f_n \to f$ almost everywhere. Show that there are sets $\{E_n\}$ such that $E := \bigcup_{n=1}^{\infty} E_n$ is conull and such that $f_n \to f$ uniformly on each $E_n$. (Compare with #4(b). Of course, you should assume that each $f_n$ and $f$ are measurable.)

6. Suppose that $f_n \downarrow f$ in $L^+$. Is it necessarily the case that

$$\int f_n(x) \, d\mu(x) \to \int f(x) \, d\mu(x)?$$

What if $\mu(X) < \infty$? What if $\int f(x) \, d\mu(x) < \infty$? What if $\int f_1(x) < \infty$?

7. Suppose that $f \in L^1(X)$. Show that for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$\int_E |f(x)| \, d\mu(x) < \epsilon$$

provided $\mu(E) < \delta$. (This is easy if $f$ is bounded.)

8. Let $f$ be a function on $[a, \infty)$ such that $f$ is bounded on bounded subsets. Recall that $f$ is improperly Riemann integrable if $f$ is Riemann integrable on each interval $[a, b]$ and

$$\lim_{b \to \infty} \int_{[a,b]} f(x) \, dm(x)$$

exists (and is finite). Show that if $f$ is nonnegative and Riemann integrable on each $[a, b]$ with $b > a$, then $f$ is improperly Riemann integrable on $[a, \infty)$ if and only if $f$ is Lebesgue integrable on $[a, \infty)$ in which case the value of the Lebesgue integral equals the value of the above limit. What happens when $f$ is not necessarily nonnegative? (“Luke, use the Monotone Convergence Theorem.”)

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1We say that $E$ is conull if $\mu(E^c) = 0$. 