Theorem 1 (Folland Theorem 2.28). Suppose that $f$ is a bounded real-valued function on $[a, b]$.

1. If $f$ is Riemann integrable, then $f$ is Lebesgue measurable (and therefore integrable). Furthermore

$$\mathcal{R} \int_a^b f = \int_{[a,b]} f(x) \, dm(x).$$  \hspace{1cm} (1)

(Henceforth, we will dispense with the notations in (1) and write simply $\int_a^b f(x) \, dx$.)

2. Also, $f$ is Riemann integrable if and only if the set of discontinuities of $f$ has measure zero.

Proof. Let $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$ and define

$$l_\mathcal{P} := \sum_{i=1}^{n} m_i \mathbb{I}_{[t_{i-1}, t_i]} \quad \text{and} \quad u_\mathcal{P} := \sum_{i=1}^{n} M_i \mathbb{I}_{[t_{i-1}, t_i]},$$

where

$$m_i := \inf \{ f(x) : x \in [t_{i-1}, t_i] \} \quad \text{and} \quad M_i := \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

Notice that

$$\int l_\mathcal{P} = L(f, \mathcal{P}) \quad \text{and} \quad \int u_\mathcal{P} = U(f, \mathcal{P}).$$

We can choose sequences of partitions $\{Q_k\}$ and $\{R_k\}$ such that

$$\lim_k L(f, Q_k) = \mathcal{R} \int_a^b f \quad \text{and} \quad \lim_k U(f, R_k) = \mathcal{R} \int_a^b f.  \hspace{1cm} (2)$$

Let $\mathcal{P}_k = \{a = t_0 < \cdots < t_n = b\}$ be a partition which is refinement of the partitions $Q_k$ and $R_k$ as well as $\mathcal{P}_{k-1}$, and which also has the property that $\|\mathcal{P}_k\| := \max(t_i - t_{i-1}) < \frac{1}{k}$. Since $\mathcal{P}_k$ is a refinement of both $Q_k$ and $R_k$, (2) holds with $Q_k$ and $R_k$ each replaced by $\mathcal{P}_k$. Since $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$, it follows that

$$l_{\mathcal{P}_{k+1}} \geq l_{\mathcal{P}_k} \quad \text{and} \quad u_{\mathcal{P}_{k+1}} \leq u_{\mathcal{P}_k}.$$
Therefore we obtain bounded measurable functions $l$ and $u$ on $[a, b]$ by

\[ l := \sup_k l_{P_k} = \lim_k l_{P_k} \quad \text{and} \quad u := \inf_k u_{P_k} = \lim_k u_{P_k}. \]

Clearly

\[ l \leq f \leq u. \]

Since bounded functions are Lebesgue integrable on $[a, b]$ and since $u = \lim_k u_{P_k}$ and $l = \lim_k l_{P_k}$, the Lebesgue Dominated Convergence Theorem implies that

\[ \int l = \mathcal{R} \int_a^b f \quad \text{and} \quad \int u = \mathcal{R} \int_a^b f. \]

Now if $f$ is Riemann integrable, the upper and lower integrals coincide and we have

\[ \int (u - l) = 0. \]

Since $u - l \geq 0$, this implies that $l = f = u$ a.e. Since Lebesgue measure is complete, $f$ is measurable, and

\[ \mathcal{R} \int_a^b f = \int f. \]

This proves the first part.

To prove the second assertion, first observe that if $x \in [a, b]$ and if $0 < \delta < \delta'$, then

\[ \sup\{ f(y) : |y - x| \leq \delta \} \leq \sup\{ f(y) : |y - x| \leq \delta' \}. \]

It follows that

\[ \lim_{\delta \to 0} \sup\{ f(y) : |y - x| \leq \delta \} = \inf_{\delta > 0} \sup\{ f(y) : |y - x| \leq \delta \}. \]

(3)

Thus we get a well defined function $H$ on $[a, b]$ by setting $H(x)$ equal to (3). Similarly, we can define $h$ on $[a, b]$ by

\[ h(x) := \lim_{\delta \to 0} \inf\{ f(y) : |y - x| \leq \delta \} = \sup_{\delta > 0} \inf\{ f(y) : |y - x| \leq \delta \}. \]

(4)

We clearly have $h(x) \leq f(x) \leq H(x)$ for all $x \in [a, b]$. 2
Suppose that \( f \) is continuous at \( x \). Then given \( \epsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( |y - x| \leq \delta \) we have \( |f(y) - f(x)| < \epsilon \). This is the same as

\[
f(x) - \epsilon < f(y) < f(x) + \epsilon.
\]

It follows from (3) and (5) that \( H(x) < f(x) + \epsilon \). Since \( \epsilon \) is arbitrary, we must have \( H(x) \leq f(x) \). Thus \( H(x) = f(x) \) in the event that \( f \) is continuous at \( x \). Similarly, combining (3) and (4) shows that \( h(x) > f(x) - \epsilon \) for any \( \epsilon > 0 \). Thus forces \( h(x) = f(x) \) when \( f \) is continuous at \( x \). In particular, \( H(x) = h(x) \) if \( f \) is continuous at \( x \).

Now suppose that \( H(x) = h(x) \). Note that the common value must be \( f(x) \). Thus given \( \epsilon > 0 \), there is — in view of (3) and (4) — a \( \delta > 0 \) such that

\[
f(x) + \epsilon = H(x) + \epsilon > \sup \{ f(y) : |y - x| \leq \delta \} \quad \text{and} \quad f(x) - \epsilon = h(x) - \epsilon < \inf \{ f(y) : |y - x| \leq \delta \}.
\]

Thus if \( |y - x| < \delta \), then we have

\[
f(x) - \epsilon < f(y) < f(x) + \epsilon \quad \text{or} \quad |f(y) - f(x)| < \epsilon.
\]

This shows that \( f \) is continuous at \( x \) if and only if \( H(x) = h(x) \).

If \( P = \{ a = t_0 < \cdots < t_n = b \} \) is any partition of \( [a, b] \) and if \( x \notin P \), then there is a \( \delta > 0 \) such that \( \{ y : |y - x| \leq \delta \} \cap P = \emptyset \). In particular, \( \{ y : |y - x| \leq \delta \} \subset (t_{i-1}, t_i) \) for some \( i \), and

\[
M_i \geq \sup \{ f(y) : |y - x| \leq \delta \}.
\]

It follows that \( u_P(x) \geq H(x) \) for all \( x \notin P \). Now let

\[
N := \bigcup_k P_k.
\]

Then \( N \) is countable, and therefore has Lebesque measure 0. Furthermore if \( x \notin N \), then

\[
u(x) := \inf u_{P_k}(x) \geq H(x).
\]

On the other hand, given \( x \notin N \) and \( \epsilon > 0 \), there is a \( \delta > 0 \) such that

\[
H(x) + \epsilon > \sup \{ f(y) : |y - x| \leq \delta \}.
\]

\(^1\)This is the first of Folland’s suggested “Lemmas”.

3
Pick $k$ such that $\frac{1}{k} < \delta$. Since $x \notin P_k$, $x \in (t_{i-1}, t_i)$ for some subinterval in $P_k$. Since $\|P_k\| < \frac{1}{k}$, $M_i \leq \sup \{ f(y) : |y - x| \leq \delta \}$ and

$$H(x) + \epsilon > u_{P_k}(x) \geq u(x).$$

Since $\epsilon$ was arbitrary, we conclude that $H(x) = u(x)$ for all $x \notin N$. In particular, $H$ is measurable and

$$\int H = \mathcal{R} \int_a^b f.$$

A similar argument implies that $h(x) = l(x)$ for all $x \notin N$. Thus $h$ is measurable and\(^2\)

$$\int h = \mathcal{R} \int_a^b f.$$

Now if $f$ is continuous almost everywhere, it follows that $H = h$ a.e. Thus the upper and lower Riemann integrals must be equal and $f$ is Riemann integrable. On the other hand, if $f$ is Riemann integrable, the upper and lower integrals are equal and

$$\int (H - h) = 0.$$

Since $H - h \geq 0$, we must have $H = h$ a.e. It follows that $f$ is continuous almost everywhere. \(\square\)

\(^2\)This is essentially Folland’s Lemma (b).