1. Let \((X,d)\) be a metric space. The open ball of radius \(r\) centered at \(x\) is defined to be 

\[ B_r(x) = \{ y \in X : d(x, y) < r \}. \]

Recall that a set \(V \subset X\) is called open if given \(x \in V\) there is a \(r > 0\) such that \(B_r(x) \subset V\). Show that open balls are open and that the set \(\tau\) of open sets in \(X\) is a topology on \(X\).

2. Suppose that \(f : [a, b] \to \mathbb{R}\) is bounded and that both \(P\) and \(Q\) are partitions of \([a, b]\). Prove that 

\[ L(f, P) \leq U(f, Q), \]

where \(L(f, P)\) and \(U(f, Q)\) are the lower and upper Riemann sums, respectively for \(f\) on \([a, b]\). (Hint: first prove the result in the case \(P = Q\), and then consider \(R := P \cup Q\).)

3. Prove that a bounded function \(f : [a, b] \to \mathbb{R}\) is Riemann integrable on \([a, b]\) if and only if for all \(\epsilon > 0\) there is a partition \(P = P_\epsilon\) of \([a, b]\) such that 

\[ U(f, P) - L(f, P) < \epsilon. \]

4. Using only the definition of Riemann integrability and the result from question 3 above, show that a continuous function \(f : [a, b] \to \mathbb{R}\) is Riemann integrable. You may make use of the theorem that implies \(f\) is uniformly continuous.

5. (Rudin: Page 31, #5) Suppose that \(f, g : (\mathcal{M}) \to [-\infty, \infty]\) are measurable functions. Prove that the sets 

\[ \{ x \in X : f(x) < g(x) \} \quad \text{and} \quad \{ x \in X : f(x) = g(x) \} \]

are measurable. (Remark: if \(h = f - g\) were defined, this problem would be much easier. Since \(-\infty - \infty\) is undefined, \(h\) is not everywhere defined.)

6. Suppose that \(Y\) is either the extended reals \([-\infty, \infty]\) or the complex numbers \(\mathbb{C}\). Let \(f_n : (X, \mathcal{M}) \to Y\) be measurable for \(n \in \mathbb{Z}^+\). Prove that the set of \(x \in X\) such that \(\lim_n f_n(x)\) exists is measurable. (Hint: use problem 5.)
7. Recall from calculus that if \( \{ a_n \} \) is a sequence of nonnegative real numbers, then \( \sum_{n=1}^{\infty} a_n = \sup_n s_n \), where \( s_n := a_1 + \cdots + a_n \). (Note that the value \( \infty \) is allowed.)

(a) Show that \( \sum_{n=1}^{\infty} a_n = \sup \{ \sum_{k \in F} a_k : F \text{ is a finite subset of } \mathbb{Z}^+ = 1, 2, 3, \ldots \} \).

(The point of this part of the problem is that if \( I \) is any set — countable or not — and if \( a_i \geq 0 \) for all \( i \in I \), then we can define
\[
\sum_{i \in I} a_i := \sup \left\{ \sum_{i \in F} a_i : F \text{ is a finite subset of } I \right\},
\]
and our new definition coincides with the usual one from calculus when both make sense.)

(b) Let \( X \) be a set and \( f : X \to [0, \infty) \) a function. For each \( E \subset X \), define
\[
\nu(E) := \sum_{x \in E} f(x).
\]
Show that \( \nu \) is a measure on \( (X, \mathcal{P}(X)) \).

(Some special cases are of note: if \( f(x) = 1 \) for all \( x \in X \), then \( \nu \) is called counting measure on \( X \). If \( f(x) = 0 \) for all \( x \neq x_0 \) and \( f(x_0) = 1 \), then \( \nu \) is called the Dirac delta measure at \( x_0 \). If \( \sum_{x \in X} f(x) = 1 \), then \( f \) is a discrete probability distribution on \( X \), and \( \nu(E) \) is the probability of the event \( E \) for this distribution.)

(c) Let \( X, f \) and \( \nu \) be as in part (b). Show that if \( \nu(E) < \infty \), then \( \{ x \in E : f(x) > 0 \} \) is countable.

(Hint: if \( \{ x \in E : f(x) > 0 \} \) is uncountable, then at least one of the sets \( \{ x \in E : f(x) > \frac{1}{m} \} \) must be infinite.)

(The result in part (c) shows that discrete probability distributions “live on” countable sample spaces.)