1. Problem #9 on page 59 of the text.

2. Problem #4 on page 73 of the text. You may use the result of problem #2 on page 73. On part (c), I found the function defined on \([e, \infty)\) by

\[ f(x) = \frac{1}{x(\log x)^2} \]

to be interesting.

3. Problem #7 on page 73 of the text. The examples \(L^p(\mathbb{R}, m), \ L^p([0, 1], m),\) and \(\ell^p\) will be important here. Note that \(\ell^p\) is the set of complex sequences \((a_i)\) such that \(\sum_{i=1}^{\infty} |a_i|^p < \infty\). Thus, \(\ell^p = L^p(\mathbb{Z}^+, \nu)\) where \(\nu\) is counting measure.

4. Let \((X, \mathcal{M}, \mu)\) be a measure space. Show that simple functions are dense in \(L^p(X, \mathcal{M}, \mu)\) for \(1 \leq p \leq \infty\).

I suggest the following strategy:

(a) Observe that it suffices to consider nonnegative \(f\).

(b) For \(1 \leq p < \infty\), apply Theorem 1.17 in the text.

(c) For \(p = \infty\), observe that if \(f\) is bounded then the \(s_i\) constructed in class (or in 1.17 in the text) not only satisfy \(s_i \nearrow f\), but \(\|s_i - f\|_{\infty} < 2^{-i}\) provided \(\|f\|_{\infty} \leq i\). (More simply said, the convergence is uniform.)

5. Let \(\mu, \nu, \) and \(\lambda\) be \(\sigma\)-finite measures on \((X, \mathcal{M})\). We'll denote the Radon-Nikodym derivative of \(\nu\) by \(\mu\) by \(\frac{d\nu}{d\mu}\).

(a) Show that if \(\nu \ll \mu\) and \(g : X \to [0, \infty]\) is measurable, then \(\int_X g \, d\nu = \int_X g \frac{d\nu}{d\mu} \, d\mu\). (As observed in class, this is a Corollary of an old Theorem.) Conclude that \(f \in L^1(\nu)\) if and only if \(f \frac{d\nu}{d\mu} \in L^1(\mu),\) and that the same formula holds.

(b) Suppose that \(\nu \ll \mu \ll \lambda\). Show that \(\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}\). Of course, “=” means “equal almost everywhere \([\lambda]\).”

(c) Suppose that \(\mu \ll \nu\) and \(\nu \ll \mu\) (we say the \(\mu\) and \(\nu\) are equivalent and write \(\nu \approx \mu\)). Show that \(\frac{d\mu}{d\nu} = \left[\frac{d\nu}{d\mu}\right]^{-1}\). Again “=” means “equal almost everywhere \([\mu]\) (or \([\nu]\))."
6. Let \( \nu \) be a complex measure on \((X, M)\).

(a) Show that there is a measure \( \mu \) and a measurable function \( \phi : X \to \mathbb{C} \) so that \( |\phi| = 1 \), and such that for all \( E \in M \),

\[
\nu(E) = \int_E \phi \, d\mu.
\] (†)

(Hint: write \( \nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4) \) for measures \( \nu_i \). Put \( \mu_0 = \nu_1 + \nu_2 + \nu_3 + \nu_4 \). Then \( \mu_0 \) will satisfy (†) provided we don’t require \( |\phi| = 1 \). Now use the “useful lemma” from lecture.)

(b) Show that the measure \( \mu \) above is unique, and that \( \phi \) is determined almost everywhere \([\mu]\). (Hint: if \( \mu' \) and \( \phi' \) also satisfy (†), then show that \( \mu' \ll \mu \), and that \( \frac{d\mu'}{d\mu} = 1 \) a.e. Also note that if \( \phi' \) is unimodular and \( E \in M \), then \( E = \bigcup_{i=1}^4 E_i \) where \( E_1 = \{ x \in E : \text{Re } \phi' > 0 \} \), \( E_2 = \{ x \in E : \text{Re } \phi' < 0 \} \), \( E_3 = \{ x \in E : \text{Im } \phi' > 0 \} \), and \( E_4 = \{ x \in E : \text{Im } \phi' < 0 \} \).

Comment: the measure \( \mu \) in question 6 is called the total variation of \( \nu \), and the usual notation is \( |\nu| \). It is defined by different methods in your text: see chapter 6. One can prove facts like \( |\nu|(E) \geq |\nu(E)| \), although one doesn’t always have \( |\nu|(E) = |\nu(E)| \); this also proves that even classical notation can be unfortunate.

7. If \( \nu \) and \( \lambda \) are complex measures on the same measurable space the we define \( \nu \perp \lambda \) and \( \nu \ll \lambda \) if the corresponding relations hold for \( |\nu| \) and \( |\lambda| \).

(a) Suppose that \( \lambda \) is a positive measure. Show that \( \nu \ll \lambda \) if and only if \( \lambda(E) = 0 \) implies \( \nu(E) = 0 \). (You’ll want to use Equation (†).)

(b) Show that if \( \nu \perp \lambda \) and \( \nu \ll \lambda \), then \( \nu = 0 \).

(c) Prove the uniqueness assertion in the Lebesgue Decomposition theorem. (Hint: start with the case where \( \mu \) and \( \nu \) are finite—so that for example, \( \mu - \nu \) is a complex measure. Then use \( \sigma \)-finiteness.)

8. Show that the \( \sigma \)-finite hypothesis is necessary in the Radon-Nikodym theorem. (Hint: let \( \nu \) be Lebesgue measure on \([0,1]\) and let \( \mu \) be counting measure (restricted to the Lebesgue measurable sets in \([0,1]\)).)