The abc conjecture asserts that if \( \epsilon > 0 \) is fixed, there are at most finitely many coprime triples \( a, b, c \) of positive integers with \( a + b = c \) and \( c^{1-\epsilon} > \text{rad}(abc) \).

The Lindemann–Weierstrass theorem asserts that if \( \alpha_1, \ldots, \alpha_n \) are distinct in \( \mathbb{A} \), then \( e^{\alpha_1}, \ldots, e^{\alpha_n} \) are linearly independent over \( \mathbb{A} \).

1. Show that the abc conjecture is false with \( \epsilon = 0 \). In fact, show that there is an infinite sequence of triples \( a_n, b_n, c_n \) of coprime positive integers with \( c_n / \text{rad}(a_n b_n c_n) \to \infty \) as \( n \to \infty \).

2. Let \( P = \{a^b : a \geq 1, b \geq 2, a, b \in \mathbb{Z} \} \). Show the abc conjecture implies that for each positive integer \( c \) there are at most finitely many solutions to \( m - n = c \), where \( m, n \in P \). (We know that the only solution to \( m - n = 1 \) with \( m, n \in P \) is \( 9 - 8 = 1 \).)

3. Using the Lindemann–Weierstrass theorem, prove that \( e \) is transcendental.

4. Using the Lindemann–Weierstrass theorem, prove that \( \pi \) is transcendental.

5. Using the Lindemann–Weierstrass theorem, prove that \( \log 2 \) is transcendental.

6. Using the Lindemann–Weierstrass theorem, prove that if \( \alpha \in \mathbb{A} \setminus \{0\} \), then \( \sin \alpha, \cos \alpha, \) and \( \tan \alpha \) are transcendental.

7. Using the Lindemann–Weierstrass theorem, prove that if \( \alpha_1, \ldots, \alpha_n \) are in \( \mathbb{A} \) and linearly independent over \( \mathbb{Q} \), then \( e^{\alpha_1}, \ldots, e^{\alpha_n} \) are algebraically independent over \( \mathbb{Q} \). (That is, if \( f \in \mathbb{Q}[x_1, \ldots, x_n] \) is nonzero, then \( f(e^{\alpha_1}, \ldots, e^{\alpha_n}) \neq 0 \).