Exercise 1. Let $f \in \mathbb{Z}[x]$ be a polynomial. For a positive integer $d$, let $N_f(d)$ be the number of solutions $n$ to $f(n) \equiv 0 \pmod{d}$. Prove that $N_f(d)$ is a multiplicative function of $d$. (Hint: Use the Chinese remainder theorem.)

Exercise 2. Suppose we wish to use our sieve argument to estimate from above the number of primes $p \leq x$ with $2p+1$ also prime. (These primes are called Sophie Germain primes after Sophie Germain, who showed in 1823 that for any such prime $p$, all integral solutions to $x^p + y^p = z^p$ have $p | xyz$.) What would be the sequence $A$ and what would be the estimate for $A_d$?

Exercise 3. Show that the number of Sophie Germain primes in $[1, x]$ is

$$O\left(\frac{x \log \log x}{(\log x)^2}\right).$$

Exercise 4. Show that the sum of the reciprocals of the Sophie Germain primes is finite.