Supplementary homework problems, due May 20, 2009

1. Suppose that for each prime p, we have an integer k_p with $0 \leq k_p < p$, $k_p = O(1)$, and such that for some real number $c > 0$,

$$\sum_{p \leq x} \frac{k_p \log p}{p} = c \log x + O(1).$$

   Prove that there is some number $C > 0$ such that

$$\prod_{p \leq x} (1 - k_p/p) \sim C/(\log x)^c \text{ as } x \to \infty.$$

2. Suppose $k_p = 2$ if p is 1 mod 4, $k_p = 0$ if p is 3 mod 4, and $k_2 = 1$. Show that this choice of numbers k_p satisfies the above problem with $c = 1$. What is the relevance of this exercise to primes of the form $n^2 + 1$?