Math 108. *Topics in combinatorics: The probabilistic method.*

Assignment 2. Due on Tuesday, 2/5/2008.

1. We are given $m = 2^{n-1}K$ sets $A_1, \ldots, A_m$, each of size n, in a universe V. Consider the following randomized algorithm for coloring: First color each point $v \in V$ randomly. Now, for each one of the sets A_i that was monochromatic after the first coloring, select a random vertex $v \in A_i$ and switch its color. Call the algorithm a failure if some set A_i originally had all but one vertex the same color and ended with all the vertices that color. Find K as large as you can (as an asymptotic function of n) so that the failure probability is less than one.
(Note that this, unfortunately, does not give us any result on $m(n)$ since there are other ways that a set A_i could end up monochromatic.)

2. Let $A_i \subseteq \Omega$, $1 \leq i \leq n$, with all $|A_i| = n$. For $\chi : \Omega \to \{-1, 1\}$ and $A \subseteq \Omega$ we will write $\chi(A) = \sum_{a \in A} \chi(a)$. Prove, for β as small as your technique allows, that there exists $\chi : \Omega \to \{-1, 1\}$ with all $|\chi(A_i)| \leq \beta$. (Use a random coloring and the large deviation results.)

3. The goal of this problem is to show that a random tournament T_n has $\text{fit}(T_n, \sigma) < Cn^{3/2}$ for all $\sigma \in S_n$, where C is a computable constant. We set $n = 2^t$ and assume (avoiding some technical stuff) that t is a positive integer. For $1 \leq i \leq t$ let $\text{fit}_i(T_n, \sigma)$ be the number of nonupsets minus the number of upsets in the games between $\sigma(j), \sigma(k)$ where

$$(2u-2)n2^{-i} < j \leq (2u-1)n2^{-i} < k \leq 2un2^{-i}$$

and $1 \leq u \leq 2^{i-1}$. (Plugging in $i = 1$ and $i = 2$ will be helpful in understanding the problem.) Call σ_1 and σ_2 i-similar if the pairs $\sigma(j), \sigma(k)$ above are the same for σ_1 and σ_2. Note that when this holds, $\text{fit}_i(T, \sigma_1) = \text{fit}_i(T, \sigma_2)$ for any tournament T on n players. This splits S_n into equivalence classes.

   (a) Give a precise formula for the number $A_i(n)$ of equivalence classes under i-equivalence.

   (b) Give precisely the distribution of $\text{fit}_i(T_n, \sigma)$ for σ fixed and T_n the random tournament.

   (c) Let i be fixed, with $n \to \infty$. Find the best constant β_i so that $A_i(n) \leq \beta_i^n$. For this β_i show that $A_i(n)\beta_i^{-n} \to 0$.

   (d) Let i be fixed. Let FAIL_i be the event that $\text{fit}_i(T_n, \sigma) > c_i n^{3/2}$ for some σ, where $c_i := \sqrt{i 2^{-i} \ln 2}$. Show that $\Pr[\text{FAIL}_i] \to 0$ as $n \to \infty$.

   (e) Show that $\Pr[\text{FAIL}_i] < 2^{-2^{i-1}}$.

   (f) Let $C := \sum_{i=1}^{\infty} c_i$. Deduce that there exists a tournament T_n on n players with $\text{fit}(T_n, \sigma) < Cn^{3/2}$ for all $\sigma \in S_n$.