Math 108. *Topics in combinatorics: The probabilistic method.*

Assignment 3. Due on Tuesday, 2/19/2008.

1. Page 37, Exercise 1.

2. Page 37, Exercise 2. Hint: take $p = n^{-1/2}$ and $l = 3n^{1/2}\ln(n)$ in the lower bound for $r(k,l)$ that we proved in class using alterations.

   (Note that these two problems give improvements on results we got in Chapter 1.)

3. Let n be even. Let $X_i = \pm 1$, $1 \le i \le n$ be chosen unifromly among all possible values with $\sum_{i=1}^{n} X_i = 0$. Let $1 \le t \le n-1$ and set $Y = \sum_{i=1}^{t} X_i$.

   (a) Compute $E[Y]$ precisely.

   (b) Computer $Var[Y]$ precisely.

   (c) For $c \in (0,1)$ fixed and $t \sim cn$, give the asymptotic value of $Var[Y]$ as $n \to \infty$.

   (d) It is known that the Y above is asymptotically normal with the mean and asymptotic variance calculated. Let T be a tournament on $n+1$ players in which each player wins precisely half his games. Consider the randomized algorithm to rank the players in which the players are considered in random sequential order, and each player is placed either above or below the previous players depending on whether he has beaten more than half of the previous players or not, respectively. Find the asymptotics of the expected value of $\text{fit}(T,\sigma)$ for the ranking σ given by this algorithm (you can leave the answer as a sum or as an integral).

4. Let $G \sim G(n,p)$ with $p = c/n$ Let X be the number of isolated triangles.

   (a) Find $E[X]$ precisely and give its limit value as $n \to \infty$ with c fixed.

   (b) Find $Var[X]$ precisely and give its limit value as $n \to \infty$ with c fixed.