Math 112 : Introduction to Riemannian Geometry
Quotients and Manifolds

Note: The following is a brief outline of some concepts we will need in class. For more details on basic topology and quotient spaces you should see Munkres' "Topology: A First Course". For information on properly discontinuous actions you can check out Boothby from our reserve list.

Let M be a set. A topology on M is a collection $\mathcal{T}$ of subsets of M (called open sets or neighborhoods) which satisfies the following properties.

1. $\emptyset, M \in \mathcal{T}$
2. if $\{U_\alpha\}_{\alpha \in J}$ is a collection of open sets then $\cup_{\alpha \in J} U_\alpha \in \mathcal{T}$; that is $\cup_{\alpha \in J} U_\alpha$ is open.
3. if $U_1, \ldots, U_k$ is a finite collection of open sets, then $\cap_{j=1}^k U_j \in \mathcal{T}$; that is, $\cap_{j=1}^k U_j$ is open.

The elements of $\mathcal{T}$ are called **open sets**. A subset $A \subset M$ is said to be **closed** (w.r.t, the topology $\mathcal{T}$) if its complement A^c is open.

A set M equipped with a choice of topology $\mathcal{T}$ is said to be a **topological space**. The topological space is said to be **Hausdorff** if for any $p \neq q \in M$ there exists open sets (i.e., neighborhoods) U and V containing p and q respectively such that $U \cap V = \emptyset$. A metric space (X, d) is a prime example of a Hausdorff topological space: in this case the topology is the collection of all sets U such that for each $p \in U$ there exists $\epsilon = \epsilon(p) > 0$ such that $B(p, \epsilon) \subset U$.

Let M be a topological space and let $\sim$ be an equivalence relation on M. We then let $M/\sim$ denote the set of equivalence classes and let $\pi : M \to M/\sim$ be the **canonical projection** which sends $p \in M$ to its equivalence class $[p] \in M/\sim$. We may then put a topology on $M/\sim$ as follows. We will say that $U \subset M/\sim$ is open if and only if $\pi^{-1}(U)$ is open. It is an easy exercise to check that this defines a topology. We call this topology the **quotient topology** (induced by $\sim$).

A map $f : X \to Y$ between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is said to be **continuous** if for any $V \subset Y$ open (w.r.t. $\mathcal{T}_Y$) the set $f^{-1}(V)$ is open in X (w.r.t. $\mathcal{T}_X$). We will say that f is a **homeomorphism** if

1. f is bijective.
2. $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous.

Now let Γ be a discrete group which acts on a topological space M via homeomorphisms. That is, for each $\gamma \in \Gamma$ we have a homeomorphism $\gamma : M \to M$ such that:

1. $e : M \to M$ is the identity map.
2. For any $\gamma_1, \gamma_2 \in \Gamma$ we have $(\gamma_1\gamma_2) \cdot x = \gamma_1 \cdot (\gamma_2 \cdot x)$.

This action defines an equivalence relation $\sim$ on M by setting $x \sim y$ if and only if there exists $\gamma \in M$ such that $\gamma \cdot x = y$. For each $x \in M$ its equivalence class $[x]$ is called an **orbit** and is denoted by Γx. We denote the set of equivalence classes by M/Γ. The action of Γ on M is said to be **free** if $\gamma \cdot x = x$ for some $x \in M$ implies $\gamma = e$. The action of Γ on M is said to be **properly discontinuous** if whenever $x, y \in M$ do not belong to the same orbit there exist open sets $U, V \subset M$ containing x and y respectively such that $\gamma \cdot U \cap V = \emptyset$ for any $\gamma \in \Gamma$.

Theorem. *Let M be a (Hausdorff) topological space and let Γ be a discrete group which acts freely and properly discontinuously on M, then M/Γ is a Hausdorff topological space when endowed with the quotient topology. Furthermore, suppose M has the structure of a C^∞-manifold and $\Gamma \leq \mathrm{Diff}(M)$, so that Γ acts via diffeomorphisms. Then M/Γ has a C^∞-structure with respect to which the canonical projection $\pi : M \to M/\Gamma$ is a local diffeomorphism.*

Example (The k-torus): Let $M = \mathbb{R}^k$ have the usual C^∞-structure and let $\Gamma = \mathbb{Z}^k$ be the group of k-tuples of integers under addition. Then Γ acts on M via translations. Indeed, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\gamma = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, then $\gamma \cdot x = x + \gamma = (x_1 + n_1, \ldots, n_k + n_k) \in \mathbb{R}^k$. If we let $\mathbb{R}^k$ have its usual C^∞-structure, we see that Γ acts via diffeomorphimsms and this action is free and properly discontinuous. By the above theorem $T^k = \mathbb{R}^k/\mathbb{Z}^k$ admits a differentiable structure such that $\pi : \mathbb{R}^k \to T^k$ is a local diffeomorphim. T^k is known as the k-torus.

Example (Real Projective Space): Consider $S^n \subset \mathbb{R}^{n+1}$ with the ususal C^∞-structure and $\Gamma = \{\mathrm{id}, A\}$ where $A : S^n \to S^n$ is the antipodal map given by $A(x) = -x$. So, Γ is isomorphic to $\mathbb{Z}_2$. One can check that Γ acts by diffeomorphisms and that the action is free and properly discontinuous. Hence, $\mathbb{R}P^n = S^n/\Gamma$ (the real projective space of dimension n) is a differentiable manifold and $\pi : S^n \to \mathbb{R}P^n$ is a local diffeomorphism.

The above considerations are important to us because of the following theorem, which we will discuss in class.

Theorem. *Let (M, g) be a Riemannian manifold and let $\Gamma \leq \mathrm{Isom}(M, g)$ be a discrete group of isometries which act freely and properly discontinuously on M. Then the smooth manifold M/Γ admits a unique Riemannian metric $\tilde{h}$ such that $\pi : (M, g) \to (M/\Gamma, h)$ is a Riemannian covering.*

This will give us a way of constructing new examples of Riemannian manifolds from old ones.