Homework Assignment #5
Due Wednesday, March 3rd.

1. In this problem, X will be a *separable* Banach space. Let B^* be the closed unit ball in X^*. We want to work out a solution to E 2.5.3 in the text. Work out your own solution, or follow the guidelines below.

(a) Show that a subset of separable metric space is separable so that we can find a countable dense subset $\{ d_k \}_{k=1}^{\infty}$ of the unit sphere $S = \{ x \in X : \|x\| = 1 \}$ in X. (Hint: a separable metric space is second countable.)

ANS: The hint gives it away. A separable metric space is always second countable: if $\{x_n\}$ is a countable dense set, then the collection of balls $\{ B_{\frac{1}{m}}(x_n) : n, m \geq 1 \}$ form a countable basis. But any subset of a second countable space is clearly second countable in the relative topology. Now observe that any second countable space is separable: just take a point in each basic open set. Since we have assumed that H is separable, it follows that S is separable.

(b) For each k, show that $m_k(\varphi) := |\varphi(d_k)|$ is a seminorm on X^* such that $m_k(\varphi) \leq 1$ on B^*.

ANS: Note that m_k is just the seminorm associated to $\iota(d_k) \in X^{**}$. It is bounded by 1 on B^* since d_k has norm 1.

(c) Show that a net $\{ \varphi_j \}$ in B^* converges to $\varphi \in B^*$ in the weak-$*$ topology if and only if $m_k(\varphi_j - \varphi) \to 0$ for all k.

ANS: Note that $m_k(\varphi_j - \varphi) \to 0$ exactly when $\varphi_j(d_k) \to \varphi(d_k)$. Thus, if $\varphi_j \to \varphi$ in the weak-$*$ topology, then $m_k(\varphi_j - \varphi) \to 0$ for all k.

Conversely, suppose that $m_k(\varphi_j - \varphi) \to 0$ for all k. This simply means that $\varphi_j(d_k) \to \varphi(d_k)$ for all k. Of course this means $\varphi_j(\alpha d_k) \to \varphi(\alpha d_k)$ for any $\alpha \in \mathbf{F}$. Let $x \in X$. If $x = 0$, then $\varphi_j(x) \to \varphi(x)$ trivially. Otherwise, let $\alpha = \|x\|$. Then given $\epsilon > 0$ there is a k such that $\|x - \alpha d_k\| < \epsilon/3$. Then we can choose j_0 such that $j \geq j_0$ implies that $|\varphi_j(\alpha d_k) - \varphi(\alpha d_k)| < \epsilon/3$. Then since φ_j and φ have norm at most one, $j \geq j_0$ implies that

$$\begin{aligned} |\varphi_j(x) - \varphi(x)| &\leq |\varphi_j(x) - \varphi_j(\alpha d_k)| + |\varphi_j(\alpha d_k) - \varphi(\alpha d_k)| + |\varphi(\alpha d_k) - \varphi(x)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Since $x \in X$ was arbitrary, $\varphi_j \to \varphi$ in the weak-$*$ topology.

(d) For each $\varphi, \psi \in B^*$, define

$$\rho(\varphi, \psi) := \sum_{n=1}^{\infty} \frac{m_n(\varphi - \psi)}{2^n}.$$

Show that ρ is a metric on B^*.

ANS: Note that $m_n(\varphi - \psi) \leq 2$, so the sum always converges to a nonnegative number. Therefore, to see that ρ is metric, it suffices to see that it is definite, symmetric and satisfies the triangle inequality.

If $\rho(\varphi, \psi) = 0$, then φ and ψ agree on $\{d_k\}$, and therefore on $\{\, \alpha d_k : k \geq 1 \text{ and } \alpha \in \mathbf{F} \,\}$. Since the latter set is dense in X, $\varphi = \psi$. Since $m_k(\varphi - \psi) = m_k(\psi - \varphi)$, we also have $\rho(\varphi, \psi) = \rho(\psi, \varphi)$. And if $\zeta \in B^*$, then we have $m_k(\varphi - \psi) \leq m_k(\varphi - \zeta) + m_k(\zeta - \psi)$ (since m_k is a seminorm). It now follows easily that $\rho(\varphi, \psi) \leq \rho(\varphi, \zeta) + \rho(\zeta, \psi)$.

(e) Show that a net $\{\, \varphi_j \,\}$ in B^* converges to $\varphi \in B^*$ in the weak-$*$ topology if and only if $\rho(\varphi_j, \varphi) \to 0$. Conclude that the topology induced by ρ on B^* is the weak-$*$ topology; that is, conclude that the weak-$*$ topology on B^* is metrizable.

ANS: Suppose that $\rho(\varphi_j, \varphi) \to 0$. Then it is easy to see that $m_k(\varphi_j - \varphi) \to 0$ for each k. Then by part (c), $\varphi_j \to \varphi$ in the weak-$*$ topology.

Conversely, suppose that $\varphi_j \to \varphi$ in the weak-$*$ topology. Then by part (c) again, $m_k(\varphi_j - \varphi) \to 0$ for each k. Let $\epsilon > 0$ be given. There is a N such that

$$\sum_{n=N-1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}. \tag{1}$$

Then, since each $m_k(\varphi - \psi)$ is bounded by 2,

$$\sum_{n=N}^{\infty} \frac{m_n(\varphi_j - \varphi)}{2^n} \leq \sum_{n=N-1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2} \quad \text{for all } j. \tag{2}$$

Now we can find j_0 such that $j \geq j_0$ implies that

$$m_n(\varphi_j - \varphi) < \frac{\epsilon}{2} \quad \text{for all } n < N.$$

Now $j \geq j_0$ implies that

$$\begin{aligned} \rho(\varphi_j, \varphi) &= \sum_{n=1}^{N-1} \frac{m_n(\varphi_j - \varphi)}{2^n} + \sum_{n=N}^{\infty} \frac{m_n(\varphi_j - \varphi)}{2^n} \\ &< \Big(\sum_{n=1}^{N-1} \frac{\epsilon}{2^{n+1}} \Big) + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\rho(\varphi_j, \varphi) \to 0$.

We have established that the topology induced by ρ is the weak-$*$ topology. In other words, the restriction of the weak-$*$ topology on the closed unit ball is metrizable.

(f) Conclude that X^* is separable in the weak-$*$ topology. (As Pedersen points out, a compact metric space is totally bounded and therefore separable.)

ANS: Since a compact metric space is separable and since the closed unit ball B^* is compact and metrizable, it is separable. If $n \geq 1$, then nB^* is just the closed n ball and nB^* is homeomorphic to B^*. Hence nB^* is separable. But $X^* = \bigcup nB^*$. Since the countable union of countable sets is countable, it follows that X^* is separable.

2. Work E 2.5.6, but use the hint from the "revised edition" of the text.

3. Suppose that H is an inner product space. Show that $|(x \mid y)| = \|x\|\|y\|$ if and only if either $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbf{F}$.

ANS: If $y = \alpha x$, then $|(x \mid y)| = |\alpha|\|x\| = \|x\|\|\alpha x\| = \|x\|\|y\|$, and similarly when $x = \alpha y$.

Now assume that $|(x \mid y)| = \|x\|\|y\|$. If $x = 0$, then $x = 0 \cdot y$. So we can assume that $\|x\| \neq 0$. Let $\tau \in \mathbf{C}$ by such that $\tau(x \mid y) = |(x \mid y)|$. Then, following the proof of the Cauchy-Schwarz inequality in our notes, for each $\lambda \in \mathbf{R}$,

$$p(\lambda) := \|\lambda\tau x + y\|^2 = \lambda^2\|x\|^2 + 2\lambda|(x \mid y)| + \|y\|^2.$$

Since $\|x\| \neq 0$, p is real quadratic polynomial. By assumption, p has zero discriminant. Hence p has a real root λ_0. Then if $\alpha_0 := \tau\lambda_0$, then $\|\alpha x + y\| = 0$ and $y = -\alpha x$.

4. Suppose that W is a nontrivial subspace of a Hilbert space H. Define the *orthogonal projection of H onto W* to be the map $P : H \to H$ by $P(h) = w$, where w is the closest element in W to h. (Alternatively, $P(h) = w$ where $h = w + w^\perp$ with $w \in W$ and $w^\perp \in W^\perp$.)

(a) Show that P is a bounded linear map with $\|P\| = 1$.

(b) Show that $P = P^2 = P^*$.

(c) Conversely, if $Q : H \to H$ is a bounded linear map such that $Q = Q^* = Q^2$, then show that Q is the orthogonal projection onto its range: $W = Q(H)$.

ANS: Let $h, k \in H$. Then h can be written uniquely as $w + w^\perp$ with $w \in W$ and $w^\perp \in W^\perp$. Similarly, $k = u + u^\perp$. But then $\alpha h + k = (\alpha w + u) + (\alpha w^\perp + u^\perp)$ and $(\alpha w + u) \in W$ while $(\alpha w^\perp + u^\perp) \in W^\perp$. Thus $P(\alpha h + k)\alpha w + u = \alpha P(h) + P(k)$, and P is linear. Since $\|h\|^2 = \|w\|^2 + \|w^\perp\|^2$, we must have $\|h\| \geq \|w\|$. That is, $\|P(h)\| \leq \|h\|$, and $\|P\| \leq 1$. Since $W \neq \{0\}$, there is a $w \in W$. Since $P(w) = w$, it follows that $\|P\| \geq 1$. Hence $\|P\| = 1$. This proves part (a).

Since $P(h) \in W$, we clearly have $P^2(h) := P\big(P(h)\big) = P(h)$, so $P = P^2$. On the other hand,

$$\big(Ph \mid k\big) = \big(P(w + w^\perp) \mid u + u^\perp\big) = (w \mid u + u^\perp) = (w \mid u) = (w + w^\perp \mid u) = \big(h \mid P(k)\big).$$

Therefore $P^* = P$. This proves part (b).

For part (c), first observe that W is closed. Suppose that $Qh_j \to h$. Then $Q^2h_j \to Qh$. Since $Q^2h_j = Qh_j$, and H is Hausdorff, $Qh = h$, and $h \in W$. Since W is closed, every $h \in H$ can be written uniquely as $w + w^\perp$ as above. Since $W = Q(H)$ and $Q^2 = Q$, it follows that $Q(w) = w$ for all $w \in W$. Thus for all $k \in H$,

$$\big(Q(h) \mid k\big) = \big(w + w^\perp \mid Q(k)\big) = \big(w \mid Q(k)\big) = \big(Q(w) \mid k\big) = (w \mid k).$$

Since k is arbitrary in H, we conclude that $Q(h) = w$ and therefore Q is the projection onto W.

5. Work problem E 3.1.9 in the text. (Remark: problem 1 implies that H is separable in the weak topology. Here we also see that, despite this, an infinite-dimensional separable Hilbert space fails to be either second countable or even first countable in the weak topology.)

ANS: Suppose that $\{e_n : n \in \mathbf{N}\}$ be a orthonormal basis for H. Let $T = \{n^{\frac{1}{2}} e_n\}$, and let C be the weak closure of T in H. The first order of business is to see that $0 \in C$. Suppose not.[1] Then there is a weak neighborhood U of 0 disjoint from T. Therefore there is an $\epsilon > 0$ and $x_1, \ldots, x_k \in H$ such that

$$U = \{\, x \in H : |(x \mid x_j)| < \epsilon \text{ for } j = 1, 2, \ldots, k \,\}.$$

Since $U \cap T = \emptyset$, for each $n \in \mathbf{N}$, we have

$$\sum_{j=1}^{k} |(n^{\frac{1}{2}} e_n \mid x_j)|^2 \geq \epsilon^2.$$

Alternatively,

$$\sum_{j=1}^{k} |(e_n \mid x_j)|^2 \geq \frac{\epsilon^2}{n} \quad \text{for all } n \in \mathbf{N}. \tag{3}$$

On the other hand, by Parseval's Identity,

$$\sum_{j=1}^{k} \|x_j\|^2 = \sum_{j=1}^{k} \sum_{n=1}^{\infty} |(x_j \mid e_n)|^2$$

[1] Ok, this is tricky. Did you come to office hours to ask about it? Why not?

which, the first sum is finite, is

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{k} |(e_n|x_j)|^2 \geq \epsilon^2 \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

by (3). This is a contradiction. Therefore we conclude that $0 \in C$ as claimed.

If H were first countable in the weak topology, then we could find a sequence $\{y_k\}_{k=1}^{\infty} \subset T$ such that $y_k \to 0$ weakly. Let Φ_y be the linear functional on H corresponding to y:

$$\Phi_y(x) := (x \mid y).$$

Recall that $\|\Phi_y\| = \|y\|$. Since convergent *sequences*[2] are bounded, for each $x \in H$,

$$\{\, |\Phi_{y_k}(x)| : k \in \mathbf{N} \,\}$$

is bounded. Therefore by the Principle of Uniform Boundedness, there is a $M > 0$ such that

$$\|y_k\| \leq M \quad \text{for all } k \in \mathbf{N}.$$

That is,

$$\{y_k\}_{k=1}^{\infty} \subset \{\, n^{\frac{1}{2}} e_n : n \leq M^2 \,\}.$$

But then $\{y_k\}$ is never in the weak neighborhood of 0 given by

$$\{\, y \in H : |(y \mid e_n)| < 1 \text{ for all } n = 1, 2, \ldots, M^2 \,\}.$$

Of course, this contradicts the assumption that $y_k \to 0$, so we can conclude that H is not (weakly) first countable and therefore H can't be metrizable in the weak topology.

6. Let H be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Show that $e_n \to 0$ weakly. Find a sequence $\{y_m\}_{m=1}^{\infty}$ of convex combinations of the e_n such that $y_m \to 0$ in norm. (This illustrates the result you proved in problem #11 on the previous homework assignment.)

ANS: Fix $x \in H$. Then $x = \sum_{n=1}^{\infty} \alpha_n e_n$, where $\alpha_n = (x \mid e_n)$. Since $\|x\|^2 = \sum_{n=1}^{\infty} \|\alpha_n|^2$, we must have $\lim_n \alpha_n = 0$. But then

$$\lim_n (e_n \mid x) = \lim_n \bar{\alpha}_n = 0,$$

and we have shown that $e_n \to 0$ in the weak topology.

Fix any $m_0 \geq 1$. Let $y_m := \frac{1}{m} \sum_{k=m_0+1}^{m_0+m} e_k$. Then $\|y_m\|^2 = \frac{1}{m^2} m = \frac{1}{m}$. Therefore $y_m \to 0$ in norm and each y_m is a convex combination of $\{e_n\}_{n \geq m_0}$.

[2]This is the whole point here. A convergent *net* need not be bounded.

7. Let $T : H \to H$ be a linear map. Show that T is bounded if and only if T is continuous when H is given the weak topology. (In the latter case, Pedersen says that T is "weak–weak" continuous. Since T is bounded exactly when it is continuous, a bounded map could be considered to be a "norm–norm" continuous map.) In fact, show that if T is "norm–weak" continuous — that is continuous as a map from H with the norm topology to H with the weak topology — then T is bounded. (Hint: use the Closed Graph Theorem.)

ANS: Suppose that T is bounded. If $x_j \to x$ weakly, then for any $y \in H$,

$$(Tx_j \mid y) = (x_j \mid T^*y) \to (x \mid T^*y) = (Tx \mid y).$$

Therefore $T_{x_j} \to Tx$ weakly and T must be weak-weak continuous.

Notice that since convergence in norm certainly implies weak convergence, a weak-weak continuous map is always norm-weak continuous. Hence it suffices to see that a norm-weak continuous operator is bounded. So assume that T is norm-weak continuous. We want to apply the Closed Graph Theorem, so suppose that $x_n \to x$ and that $Tx_n \to y$ in *norm*. By assumption, $Tx_n \to Tx$ weakly. Since we also have $Tx_n \to y$ weakly and since the weak topology is Hausdorff, we must have $y = Tx$. Thus T is bounded (by the Closed Graph Theorem).

8. Prove Lemma 88. Thus, if $x, y \in H$, then define $\theta_{x,y}$ to be the rank-one operator $\theta_{x,y}(z) = (z \mid y)x$. Also define $B_f(H) = \{\, \theta_{x,y} : x, y \in H \,\}$. Then if $T \in B(H)$,

(a) $T\theta_{x,y} = \theta_{Tx,y}$ and $\theta_{x,y}T = \theta_{x,T^*y}$,

(b) $\|\theta_{x,y}\| = \|x\|\|y\|$,

(c) $\theta^*_{x,y} = \theta_{y,x}$,

(d) $T \in B_f(H)$ if and only if $\dim T(H) < \infty$, and

(e) $B_f(H)$ is a $*$-closed, two-sided ideal in $B(H)$.