Homework Assignment #5
Due Wednesday, March 3rd.

1. In this problem, X will be a *separable* Banach space. Let B^* be the closed unit ball in X^*. We want to work out a solution to E 2.5.3 in the text. Work out your own solution, or follow the guidelines below.

(a) Show that a subset of separable metric space is separable so that we can find a countable dense subset $\{ d_k \}_{k=1}^{\infty}$ of the unit sphere $S = \{ x \in X : \|x\| = 1 \}$ in X. (Hint: a separable metric space is second countable.)

(b) For each k, show that $m_k(\varphi) := |\varphi(d_k)|$ is a seminorm on X^* such that $m_k(\varphi) \leq 1$ on B^*.

(c) Show that a net $\{ \varphi_j \}$ in B^* converges to $\varphi \in B^*$ in the weak-$*$ topology if and only if $m_k(\varphi_j - \varphi) \to 0$ for all k.

(d) For each $\varphi, \psi \in B^*$, define

$$\rho(\varphi, \psi) := \sum_{n=1}^{\infty} \frac{m_n(\varphi - \psi)}{2^n}.$$

Show that ρ is a metric on B^*.

(e) Show that a net $\{ \varphi_j \}$ in B^* converges to $\varphi \in B^*$ in the weak-$*$ topology if and only if $\rho(\varphi_j, \varphi) \to 0$. Conclude that the topology induced by ρ on B^* is the weak-$*$ topology; that is, conclude that the weak-$*$ topology on B^* is metrizable.

(f) Conclude that X^* is separable in the weak-$*$ topology. (As Pedersen points out, a compact metric space is totally bounded and therefore separable.)

2. Work E 2.5.6, but use the hint from the "revised edition" of the text.

3. Suppose that H is an inner product space. Show that $|(x \mid y)| = \|x\|\|y\|$ if and only if either $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbf{F}$.

4. Suppose that W is a nontrivial subspace of a Hilbert space H. Define the *orthogonal projection of H onto W* to be the map $P : H \to H$ by $P(h) = w$, where w is the closest element in W to h. (Alternatively, $P(h) = w$ where $h = w + w^\perp$ with $w \in W$ and $w^\perp \in W^\perp$.)

(a) Show that P is a bounded linear map with $\|P\| = 1$.

(b) Show that $P = P^2 = P^*$.

(c) Conversely, if $Q : H \to H$ is a bounded linear map such that $Q = Q^* = Q^2$, then show that Q is the orthogonal projection onto its range: $W = Q(H)$.

5. Work problem E 3.1.9 in the text. (Remark: problem 1 implies that H is separable in the weak topology. Here we also see that, despite this, an infinite-dimensional separable Hilbert space fails to be either second countable or even first countable in the weak topology.)

6. Let H be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Show that $e_n \to 0$ weakly. Find a sequence $\{y_m\}_{m=1}^\infty$ of convex combinations of the e_n such that $y_m \to 0$ in norm. (This illustrates the result you proved in problem #11 on the previous homework assignment.)

7. Let $T : H \to H$ be a linear map. Show that T is bounded if and only if T is continuous when H is given the weak topology. (In the latter case, Pedersen says that T is "weak–weak" continuous. Since T is bounded exactly when it is continuous, a bounded map could be considered to be a "norm–norm" continuous map.) In fact, show that if T is "norm–weak" continuous — that is continuous as a map from H with the norm topology to H with the weak topology — then T is bounded. (Hint: use the Closed Graph Theorem.)

8. Prove Lemma 88. Thus, if $x, y \in H$, then define $\theta_{x,y}$ to be the rank-one operator $\theta_{x,y}(z) = (z \mid y)x$. Also define $B_f(H) = \{\, \theta_{x,y} : x, y \in H \,\}$. Then if $T \in B(H)$,

(a) $T\theta_{x,y} = \theta_{Tx,y}$ and $\theta_{x,y}T = \theta_{x,T^*y}$,

(b) $\|\theta_{x,y}\| = \|x\|\|y\|$,

(c) $\theta_{x,y}^* = \theta_{y,x}$,

(d) $T \in B_f(H)$ if and only if $\dim T(H) < \infty$, and

(e) $B_f(H)$ is a $*$-closed, two-sided ideal in $B(H)$.