Consider $[-1, 1]$ integration. We'll fix $n=2$, i.e., 3 nodes.

Choosing nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, we had Newton-Cotes quadrature

$$Q_2(f) = \sum_{k=0}^{2} w_k f(x_k)$$

with $w_0 = w_2 = \frac{1}{3}$, $w_1 = \frac{4}{3}$

This integrates degree-2 polynomials exactly:

$$\int_{-1}^{1} f(x) \, dx$$

a) Explain why this happens to integrate $x^m$ for $m$ odd, exactly, too.

Now allow nodes to move inwards from $\pm 1$, i.e., $x_0 = -\alpha$, $x_1 = 0$, $x_2 = \alpha$.

and choose $w_0 = w_2 = \beta$

b) Use degree-0 exact integration to fix $w_1$.

c) Write exactness conditions for degree-2:

and for degree-4:

d) Solve these for $\alpha$, $\beta$

e) Up to what polynomial order is integrated exactly now?
MATH 116 WORKSHEET: simple Gaussian quadrature

SOLUTIONS

Consider $[-1, 1]$ integration. We'll fix $n=2$, i.e. 3 nodes.

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$$Q_2(f) = \sum_{k=0}^{2} w_k f(x_k) \quad \text{with} \quad w_0 = w_2 = \frac{1}{3}, \quad w_1 = \frac{4}{3}$$

This integrates degree-2 polys exactly:

$$\begin{array}{c|c|c|c|c|c}
 & \frac{1}{3} & \frac{1}{3} & \frac{4}{3} \\
\hline
-1 & 0 & 1 \\
\end{array}$$

a) Explain why this happens to integrate $x^m$ for $m$ odd, exactly, too.

$$\int_{-1}^{1} x^m \, dx = 0 \quad \text{for} \quad m \text{ odd}, \quad \text{&} \quad \frac{1}{2}(-1)^m + \frac{2}{3}0^m + \frac{1}{2}1^m = 0 \implies \text{exact!}$$

Now allow nodes to move inwards from $\pm 1$, i.e. $x_0 = -\alpha$, $x_1 = 0$, $x_2 = \alpha$, and choose $w_0 = w_2 = \beta$.

b) Use degree-0 exact integration to fix $w_1$:

$$\beta + w_1 + \beta = \int_{-1}^{1} 1 \, dx = 2$$

$w_1 = \int_{-\alpha}^{\alpha} x \, dx = \frac{1}{2}\alpha^2$.

c) Write exactness conditions for degree-2:

$$\beta \alpha^2 + \beta 0^2 + \beta 0^2 = \int_{-1}^{1} x^2 \, dx = \frac{\alpha^2}{3}$$

and for degree-4:

$$\beta \alpha^4 + 0 + 0 = \int_{-1}^{1} x^4 \, dx = \frac{\alpha^4}{5}$$

d) Solve these for $\alpha, \beta$ divide:

$$\alpha^2 = \frac{3}{\beta}, \quad \alpha = \sqrt{\frac{3}{\beta}}$$

so $2\beta \cdot \frac{3}{\beta} = \frac{2}{3}, \quad \beta = \frac{3}{4}, \quad w_1 = \frac{3}{4}$.

e) Up to what polynomial order is integrated exactly now?

$p \in P_5$ is exact. Note $5 = 2n+1$ ($n=2$), since you did $0, 2/3, \frac{3}{4}$, and all odd are exact.