Course: Crash course in selected parts of PDEs. And focusing on boundary integral equations and global approximation.

Syllabus:
- HW: weekly, web-only, books.
- Topics: project (go further w/ submitting or apply to problems, or new PDE).
- Grading: HW + final project roughly equal. No exams.
- Textbooks: To be announced.
- Impact of numerical computation algorithms.

PDEs in this course:
1) Laplace eqn. $\Delta u = 0$
   - BVP: $\Delta u = 0$ in $\Omega$
   - $\frac{\partial u}{\partial n} = g$ given
     - or in exterior region $\mathbb{R}^d \backslash \bar{\Omega}$, e.g. $\nabla u \to 0 \ (\text{in } \mathbb{R}^d)$ as $|x| \to \infty$
   - Applications: Electrostatics, potential $\Phi = \phi(x)$, $\phi(x) = \frac{F}{x}$, conformal mapping $U = \phi(x)$
   - harmonic function $u$.

2) Helmholtz eqn. $(\Delta + k^2) u = 0$ $k$ = wavenumber.
   - Wave eqn: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $c$ = wave speed.
   - Acoustics: $u(x,t) = \psi(x) e^{-iwt}$ $t$ sub. into WE; $\frac{\partial u}{\partial t} = (-iw) \psi$
   - Scattering $\psi(x) e^{-iwt}$ $u(x,t) = 0$ $u(x) \text{ for } x \in \mathbb{R}^d \backslash \bar{\Omega}$
   - Elasticity: $E \frac{\partial^2 u}{\partial x^2} + \sigma = 0$, $\sigma = \text{stress}$
   - Translation $\frac{\partial u}{\partial x}$.

Eigenvalue prob. $\begin{cases} (\Delta + \lambda) u = 0 \text{ in } \Omega \text{ compact domain in } \mathbb{R}^d \end{cases}$
- Eigenfunctions $\psi_j$, $\lambda_j$.
- Applications: acoustics, quantum mechanics.

Missing: Heat eqn., Stokes eqn. (fluids), Navier-Stokes. (weak fluids)

Overall approach: push problems to the boundary $\partial \Omega$. 

10/24/06
to get to PDE we'll need:
1) algebra
2) almost all run, PDE boils down to basis
3) bit of ordinary diff. eqn.

Numerical Lin. Alg.

Lin Alg recap:
- \( A \in \mathbb{R}^{m \times n} \)
- \( Ax = b \)
- \( \text{row space} \) of \( A \): \( \text{col} \) of \( A \)
- \( \text{null space} \) of \( A \): \( \text{null} \) of \( A \)
- \( \text{rank} \) of \( A \): \( \text{dim} \) of \( \text{col} \) of \( A \)

Say \( A \) has full rank: what needs to hold so \( A \) is one-to-one?
- \( \text{rank} \) of \( A \) must be \( \text{dim} \) of \( \text{col} \) of \( A \)

Then: \( (m \times n) \)
- \( A \) has full rank \( \iff \) \( \text{map} \) is 1-1 (sols. to \( Ax = b \) unique if exists)

Square \( (n \times n) \):
- \( A^{-1} \) exists s/t \( AA^{-1} = A^{-1}A = I \)
- then solns. \( x = A^{-1}b \) is unique vec. of solns. of expansion of \( b \) in basis of cols. of \( A \)

Application: polynomial approximation
- Let \( \{ x_1, x_2, \ldots, x_n \} \) be numbers
- Claim: \( A = [x_i^{j-1} \mid i=1, \ldots, n; j=1, \ldots, n+1] \) is nonsingular

How does \( A \) arise? Say have data \( (x_j, y_j) \) \( j=1, \ldots, n \) points in plane
- What is \( n^{\text{th}} \) degree polynomial passing through data?
- \( \hat{p}_n(x) = \sum_{i=0}^{n} c_i x^i \)
- Lin. eqns: \( \hat{p}_n(x_j) = y_j \) \( j=1, \ldots, n \)

\( \begin{cases} c_0 + c_1 x_1 + \cdots + c_n x_1^n = y_1, \\ c_0 + c_1 x_2 + \cdots + c_n x_2^n = y_2, \\ \vdots \\ c_0 + c_1 x_n + \cdots + c_n x_n^n = y_n. \end{cases} \)

Suppose \( \mathbf{z} \neq \mathbf{z}' \) were 2 such solutions:
Then \( \hat{p}_n(x) = \hat{p}_n(x) \) is nontrivial degree \( (n-1) \) poly, which must vanish at each \( x_j \), i.e. have \( n \) distinct roots: impossible, so \( \mathbf{z} \) is unique = \( A \) full rank.


dim. (square non-
A = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \quad \text{rank}(A) = 2, \quad \text{null}(A) = \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ 0.1 \\ 1 \end{bmatrix} \\
\text{Vandemonde:} \quad A = \text{vander}(x); \quad \text{check, hard to view #s.} \\
\text{aray of show } A: \quad \text{image}(A), \quad \text{colorbar,} \\
\text{spy}(A), \quad \text{plot}(A) \quad \text{graphs each col.} \quad \text{plot}(x, A) \quad \text{looks correct x-values} \\
\text{flip } A \text{ in } j \text{ axis: } \quad A = AC(:, \text{end} : -1 : 1); \\
\text{rank}(A) = 21. \quad \text{not } 30. \quad -1 : 0.07 : 1. \\
oh, \text{try } n = 40: \quad \text{rank}(A) = 36. \quad \text{why?} \quad \text{numerical rank.} \\
\neq \text{theoretical rank.} \\
\text{Start Lec. 2.} \\
\text{Need more theory. : orthogonality.} \\
A^* \text{ hermitian transpose.} \quad (A^*)^* = A^{*^*}; \\
\text{inner prod. : } x^* y = \sum_{i=1}^n x_i y_i; \quad (AB)^* = B^* A^* \quad (A^*)^* = (A^{-1})^* \quad \text{row vecs.} \\
\text{vectors have } 0 \|x\| = \|x\| \Rightarrow x = 0 \quad 2-norms also \|xy\| \leq \|x\| \|y\| \text{ tri.} \quad \text{Cauchy-Schwarz.} \\
\text{orthog.: } x^* y = 0. \\
\text{Thus: vectors in an orth. set are LI \quad (prove it)} \\
\text{in an orth. vecs. in } C^n \text{ form basis: if unit length, an ornb.} \\
\text{Q - ornb. , stack in cols of } Q, \quad \text{then } Q^{-1} = Q^* \quad \text{ie } Q \text { unitary (real-valued).} \\
\text{why?} \quad (Q^* Q)_{ij} = \sum_{i=1}^n q_j^* q_i = q_j^* q_i = \delta_{ij} \\
\text{so } Q^* Q = I \\
\text{so } \text{Q is a basis of expansion of } b \in \text{or nb. } b = \sum_{i=1}^n q_i^* b_i \quad \text{no inverse rel to nice.} \\
\|Q x\| = \sqrt{(Q x)^* (Q x)} = \sqrt{x^* Q^* Q x} = \|x\| \quad \text{so } Q \text{ transformation preserves lengths.} \\
\text{if det } Q = 1, Q \text{ real, its rigid roto.} \\
\text{Matrix has 2-norms too! - guess meaning?} \\
\|A\| \text{ is smallest number } C \text{ s.t. } \|A x\| \leq C \|x\| \forall x \in C^n \text{ is max growth factor of a vector.} \\
\|A\| = \sup_{x \neq 0} \frac{\|A x\|}{\|x\|}; \text{ use 2-norms, matrix norm induced by vec 2-norms.} \\
\text{what is 2-norm of org. matrix } (a_{ij}) \quad \text{max } |a_{ij}|
rank-1 matrix $A = uv^*$

- why is $\text{rank}(A) = 1$?
- compute 2-norm: $\|Ax\| = \|uv^*x\| = \|v^*x\| \|u\|$ is equality? 
  - yes, $x = v$.

Submultiplicative: \( \|AB\| \leq \|A\| \|B\| \)

why?

Why?

Thm 9.1: $\|QA\| = |A|$ (unitary from left, preserves any norm)

pf: $\|QA\| = \|A\|$ unit.

true from right?

$\|AB\| \leq |A| \|B\| = |A|$ (fmt)

Sing Val Decomp.

- as important as spectral decmp but few know it!

geometric: every matrix $A \in \mathbb{C}^{m \times n}$ maps unit ball into a hyperellipsoid.

\[ V_1, V_2 \xrightarrow{A} \sigma_1 u_1, \sigma_2 u_2 \]

There is full rank \( r \in \mathbb{N} \)

left sing. vecs $u_1, \ldots, u_r$ unit vecs along ellipsoid axes (are orthog.)

sing. values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$

right sing. vecs $v_1, \ldots, v_n$ are projections of $\sigma_j u_j$. (amazingly, also orthog.)

If $\text{rank}(A) = r$, $\sigma_1, \ldots, \sigma_r > 0$, while $\sigma_{r+1} = \ldots = \sigma_n = 0$

algebra:

\[ Av_j = \sigma_j u_j \quad j = 1, \ldots, n \]

\[ A \left[ \begin{array}{c} u_1 \\ \vdots \\ u_r \end{array} \right] = \left[ \begin{array}{c} \sigma_1 u_1 \\ \vdots \\ \sigma_r u_r \end{array} \right] \]

\[ V = \left[ \begin{array}{c} u_1 \\ \vdots \\ u_r \\ \eta \end{array} \right] \quad \hat{V} = \left[ \begin{array}{c} \sigma_1 u_1 \\ \vdots \\ \sigma_r u_r \\ 0 \end{array} \right] \]

\[ A = U \hat{V} \]

usual to complete $\hat{V}$ approx $V$ ortog.

in which case $A = U \Sigma V^*$

Defn. SVD: $A = U \Sigma V^*$

If can prove every $A$ has SVD, will show: every matrix (complex, once) is

rotation $\rightarrow$ stretching $\rightarrow$ rotation

ie: every matrix is diagonal when expressed in correct basis for $\mathbb{R}^m \times \mathbb{R}^n$.

Cf: eigenvalue decmp. $A = V D V^{-1}$ which only for square, $A$ regular (full set of evs)
If $A$ square & invertible, $A^{-1} = (UΣV^*)^{-1} = VΣ^-1U^*$
so $A^{-1}$ has same SVD as $A$ except $S$ is diagonal entries $S_i^-$

Worksheet — need $Σ_1 = ||A||_2$, $||A||_2 = ||A^*||_2$,

Proof of Existence of SVD: Skip (grade read).

Define $S_i = ||A||_2$. $||A||_2$ opt so sup $||Ax||_2$ achieved somewhere, call it $v_1$, $||v_1||_2 = 1$
& $Av_1 = S_1 u_1 = define u_1, ||u_1||_2 = 1$

Extend $v_1$ to orid for $C^n$ : $v_1^3$ stuck in col $V$, unit
$V_1^3 1$

Calc $U_i^*AV_i = S = \begin{bmatrix} S_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_n \end{bmatrix}$ where $w$ is some vec in $C^{n-1}$

Since $Av_1 = u_1, u_2, u_3,$

Bound $||S||$ by $\| \begin{bmatrix} S_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & S_n \end{bmatrix} \| = \| \begin{bmatrix} S_1^2 + \|w\|^2 \\ \vdots \\ S_n^2 + \|w\|^2 \end{bmatrix} \|$
$\geq \sqrt{S_1^2 + \|w\|^2 + \cdots + S_n^2 + \|w\|^2}$

But since $U, V$ unitary, $||S|| = ||A|| = S_1$ by Thm. 3.1

Induction: $n-1 \Rightarrow A$ has SVD trivially.

Now prove if $B$ has SVD then $A$ has one: $A = USV^* = U [U_2 \cdots U_n] [S_1 \cdots S_n] V^*$

SVD for $A$.qed.

Anatomy of SVD:

- $r = \text{rank } A = \# \{ \sigma_j : \sigma_j > \varepsilon \}$ since $U, V$ full rank.

- $\det A = \prod_{j=1}^r \sigma_j$ (prove it)

- Numerical rank $r_\varepsilon = \# \{ \sigma_j : \sigma_j > \varepsilon \}$ where $\varepsilon$ is tolerance related to rounding errors in CPU.

- Generally $\varepsilon = \delta_i \cdot \text{(relative roundoff)}$