Plan:
defn. S, D.
change of var.
jump relations.
DGF formulation
cont. homolog.

Single-layer operator: given body force \( F \),
\[
S(x) := \int_{\partial D} \phi(x,y) \psi(y) \, dy,
\]
int. op.
interpret: \( \phi \) - charge density
\( \psi(x) \) - potential due to charge \( \phi \)
(recall charge density \( \psi \) on \( \partial \Omega \), body forces \( \phi \).

Double-layer operator: given \( T \)
\[
T(x) := \int_{\partial D} \phi(x,y) \psi(y) \, dy,
\]
int. op.
interpret: \( \phi \) - charge density
\( \psi(x) \) - potential due to charge \( \phi \)
(recall charge density \( \psi \) on \( \partial \Omega \), body forces \( \phi \).

GRF state, \( x \in \Omega \), \( u(x) = \int_{\partial D} \phi \psi \), \( \phi \), \( \psi \) from body forces.

Cor 1. Choose \( L = \partial \Omega \), then \( u(x) = \operatorname{mean} \int_{\partial D} \phi \psi \) on \( \partial \Omega \).

2. Max-min of harmonic function must occur on \( \partial \Omega \) (unless \( u = \text{const} \)).
pf: suppose max at \( x \in \Omega \), \( \exists \phi \psi \) with \( r > 0 \), but \( u(x) = \text{mean over } \partial \Omega \), contradiction, unless \( u = \text{const.} \)

3. Dirichlet EVP \( \Delta u = 0 \) in \( \Omega \), \( u = f \) on \( \partial \Omega \), has at most 1 soln.
pf: suppose \( u, v \) solns., then \( u-v = 0 \) on \( \partial \Omega \), by max principle \( u-v = 0 \) in \( \Omega \).

Recall uniqueness of \( u \) at approxf. \( \Omega \), \( \partial \Omega \), show \( u = S \phi \) for \( \phi \) = 1, has constant h, k-h in \( \Omega \), \( \phi \) cont. \( \Omega \), \( \partial \Omega \), due to surface charge.

Then, let \( \Omega \) be \( C^1 \) cont. (ie \( \psi \) \in \( C^1 \)), \( \phi \in \( C^2 \)), \( u = S \phi \).

Then \( u \) cont. in \( \mathbb{R}^2 \); for \( x \in \Omega \), \( u(x) := \int_{\partial \Omega} \phi \psi \, dy \), exists as improper int.
Theorem 1. If \( D \subset \mathbb{R}^2 \) is a simply connected domain and if \( \mathbf{F} : D \to \mathbb{R}^2 \) is a smooth vector field on \( D \), then the line integral of \( \mathbf{F} \) around any closed curve \( \Gamma \) in \( D \) is zero.

Proof. Let \( \Gamma \) be a closed curve in \( D \). By the Fundamental Theorem of Line Integrals, the line integral of \( \mathbf{F} \) around \( \Gamma \) is equal to the flux of \( \mathbf{F} \) through the area enclosed by \( \Gamma \).

Flux of \( \mathbf{F} \) through \( \Gamma \):

\[
\Phi(\mathbf{F}) = \iint_D \mathbf{F} \cdot \mathbf{n} \, dA
\]

Since \( D \) is simply connected, \( \Phi(\mathbf{F}) = 0 \) by the property of simply connected domains.

Thus, the line integral of \( \mathbf{F} \) around any closed curve \( \Gamma \) in \( D \) is zero.

\[ \square \]
\( \mathcal{R}(ii) \) is integral eqn in \( C^2(\Omega) \) for \( \tau \) given \( v \) (boundary values approaching from inside):

\[
(I - 2D)\tau = -2v - 2^{nd}\text{kind IE}
\]

Thus, if \( \mathcal{R}(i) \) has soln \( \tau \) in \( (I - 2D)\tau = -2f \), then \( \tau = Dv = \Delta v = 0 \) in \( \Omega \)

(proof for all \( \tau \) solns & interior Dirichlet BVP) \( \tau = f \) on \( \partial \Omega \)

(follows from \( \mathcal{R}(ii) \))

Numerical method: While solving \( \text{(x)} \) to get \( \tau \) at nodes, then use these nodes to approx. \( v \)

Adv: reduced 2nd to 1st problem! \( v \) small lin. system, don't need to compute all \( \tau \) unless want.

Disadv: 

We can say more: sch. kernel of \( D \) is continuous for \( C^2 \) domains for \( C^2 \) domains \( \Rightarrow D \) compact op. 

Intuitively:

Continuity of \( \frac{\partial \phi(x)}{\partial x} \) only

Proof:

\[
y(t), t \in [0, 2\pi]
\]

parametric \( \tau \)

\( y(t) \in \mathbb{R}^3 \)

If \( \Omega \) is \( C^2 \), means \( \tilde{y}(t) = \begin{pmatrix} \tilde{y}_1(t) \\ \tilde{y}_2(t) \\ \tilde{y}_3(t) \end{pmatrix} \) continuous (bounded) vector frame.

Also demand \( \tilde{y}(t) > 0 \) \( \forall t \) - speed nonvanishing.

\( D \)'s kernel \( k(s,t) = \frac{1}{2\pi} \frac{\tilde{y}(t) \cdot (y(s) - y(t))}{\|y(s) - y(t)\|^2} \) (last line): cont. for \( s \neq t \)

\( \lim_{s \to t} k(s,t) \) need l'Hopital rule twice:

\[
\frac{\partial}{\partial t} \tilde{y}(t) = \begin{pmatrix} \tilde{y}_1'(t) \\ \tilde{y}_2'(t) \\ \tilde{y}_3'(t) \end{pmatrix}, \quad \frac{\partial}{\partial s} \tilde{y}(t) = \begin{pmatrix} \tilde{y}_1'(t) \\ \tilde{y}_2'(t) \\ \tilde{y}_3'(t) \end{pmatrix}
\]

\[ \lim_{s \to t} k(s,t) = \tilde{y}(t) \cdot \tilde{y}'(t) \]

[Finish & debug sigs from 2006 notes?]
Double-layer given as \( \Phi \), \( \Phi \) in \( H^2 \), \( \int_{\partial \Omega} \Phi \) is unique in \( \mathbb{R} \).

Thus \( \Phi = \mathcal{D} \mathcal{L} \) solves \( \text{BVP} \), so is the unique solution.

Name, math: \( \text{N trick on } \Phi \) to get \( \mathcal{L} \) at nodes, then use same tools to approximate \( \Phi \) for all \( \xi \).

Adv: reduced Ad to Ad prob, much fewer degrees of freedom (size of linear system).

Disadv: linear sys is dense, direct discretization of BVP gives large sparse, system.

Can say more about \( \mathcal{L} : C(\Omega) \to C(\partial \Omega) \) : This kernel continuous for \( C^2 \) domains.

Why cont? - continuity of \( \frac{\partial \Phi(y)}{\partial y} \) are circles passing through \( y \), tangent to \( \partial \Omega \).

If \( \partial \Omega \) has constant curvature, \( \mathcal{L} \) approx. \( y \) on one of these.

kernel of \( \mathcal{L} \) is \( k(x,t) = \frac{1}{4\pi} \frac{\delta(t) \cdot (y(t) - y(x))}{|y(t) - y(x)|^2} \)

\( \lim_{x \to t} k(x,t) \) top & bottom vanish \( \Rightarrow \) Hopital: \( \frac{dx}{dt} \text{ top} = \delta(t) \cdot y(t) \to 0 \) \( \Rightarrow \).

\( \frac{dy}{dt} \text{ bottom} = 2y(t) \cdot (y(t) - y(x)) \), \( \frac{dx}{dt} \text{ bottom} = 2y(t)^2 \)

So \( \lim_{x \to t} k(x,t) = \frac{1}{4\pi} \frac{\delta(t) \cdot y(t)}{|y(t)|^2} = -\frac{K(x)}{4\pi} \)

\( K = \text{curvature} \geq 0 \) (convex).

Need for \( k(x,t) \) in Nyström.
Thus: $I - 2D$ is injective, i.e. trivial nullspace, i.e. $I$ not equal to $2D$.

Cor: by Fredholm alternative in $C(\Gamma_0)$, $(I - 2D)u = f$ has unique soln. $u$.

- soln to BVP exists $\forall f \in C(\Gamma_0)$.
- Normally, such BVPs, Fredholm alt., were first such proofs.

**Other BVPs for Laplace eqn:**

1. *Dirichlet:
   \[
   \begin{align*}
   \Delta u &= 0 \text{ in } \Omega, \\
   u &= f \text{ on } \partial \Omega \\
   \end{align*}
   \]
   - has unique soln. $\forall f$.
   - Follows from $\hat{u}(x) = \hat{f}(x)$, "Kelvin transform of $u$."
   - Also being harmonic in $\Omega^\circ = \{ x : x \in \hat{\Omega} \setminus \partial \Omega \}$.
   - Interior problem, unique, exists.

2. *Neumann:
   \[
   \begin{align*}
   \Delta u &= 0 \text{ in } \Omega, \\
   \frac{\partial u}{\partial n} &= g \text{ on } \Gamma \\
   \end{align*}
   \]
   - has non-unique soln. if $u$ soln, so is $u + c$.

   \[
   \int_{\Delta} g \, dA = 0 \\
   \text{terms out sufficient condition for soln. to exist}
   \]

3. *Robin:
   \[
   \begin{align*}
   \Delta u &= 0 \text{ in } \Omega, \\
   \frac{\partial u}{\partial n} + \gamma u &= g \text{ on } \Gamma \\
   \end{align*}
   \]
   - has unique soln. $\forall f$. but not always.

eg. Folland PDE book.

**Harmonic Waves:**

- *Helmholtz eqn.* $\Delta + k^2 u = 0$
- *Wave eqn.* $u = \frac{1}{2} \frac{c}{k} z$, $\phi = u - c t$, $\omega \sqrt{\Delta}$
- *Schrödinger eqn.* $i \hbar \frac{\partial u}{\partial t} + \frac{1}{2m} \Delta u = \frac{\hbar^2}{2m} \psi^2$

\[k = \text{wavenumber} = \frac{2\pi}{\text{wavelength}}.\]

\[u = \text{wavefront.}\]

\[\phi = \text{wavepacket.}\]

\[\rho = \text{waveparticle.}\]

\[\rho = \text{wavefunction.}\]

**Interior BVP**

\[
\begin{align*}
\Delta u &= 0 \text{ in } \Omega, \\
\text{driving a cavity on its body.}
\end{align*}
\]

- unique solution unless \( \Delta u_1 = \text{Dirichlet in } \Omega \) has non-trivial soln, i.e. $k^2 \lambda$ is eigenvalue of $-\Delta$ w/ Dirichlet BCs.

**Consequence:** $\Delta$ compact. $E_k$ discrete. $k_1$ limit pt. is $\infty$.

- we'll return to these Dirichlet eigenvalues later.
Exter. BVP:
\[ (\Delta + k^2) u \mathbf{z} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \mathcal{S}, \quad d=2,3, \ldots \]
\[ u \mathbf{z} = f \quad \text{on} \quad \partial \mathcal{S} \]

\[ u \mathbf{z} = \mathcal{F}(\mathbf{r}) = \sum_{\lambda} a \mathbf{\Phi}_\lambda \mathbf{e}^{-ik \mathbf{r} \cdot \mathbf{\lambda}} \quad \text{for} \quad d=2 \]

\[ \text{has unique soln.} \quad \forall \mathbf{r} \in \mathcal{S}. \quad \text{(proof OK Thm 3-7)} \]

Scattering of waves: say incident wave \( u^i : \mathbb{R}^2 \to \mathcal{C} \) satisfies \( (\Delta + k^2) u^i = 0 \) in \( \mathbb{R}^2 \).

Then if \( f = -u^i \mathbf{\nabla} \cdot \mathbf{z} \), \( u = u^i + u^s \) solves \( (\Delta + k^2) u = 0 \) in \( \mathbb{R}^2 \setminus \mathcal{S} \).

\[ u = 0 \quad \text{on} \quad \partial \mathcal{S} \]

with obstacle generating only outgoing waves.

Solving Helmholtz BVP:
Fundamental soln.
\[ \mathcal{F}(x,y) = \frac{1}{4\pi} \mathcal{H}_0^{(1)}(k|x-y|) \quad d=2. \]

\text{outgoing Hankel func, a special func, MATLAB/mompy can evaluate.}

As \( r=|x-y| \to 0 \), \( \mathcal{F}(x,y) = -\frac{i}{2\pi} \ln|x-y| + O(1) \), i.e. same sing. as for Laplace's eqn.

\[ \Rightarrow \quad \text{All jump relations same as before} \]

\text{Take-home msg: can replace Laplace & Helmholtz & BIEs same as before! (HW6)}

When Hankel from?
\[ u(x,y) = f(kr) e^{i\mathbf{\epsilon} \cdot \mathbf{r}} \quad \text{sep. of var.; fix } \mathbf{\epsilon} \in \mathbb{C}, \text{ & find } f(\mathbf{\epsilon}) \text{ sat. Helmh. Eqn.} \]

0: \( k^2 \mathbf{\nabla}^2 u + k^2 u = f \quad \text{Bessel's eqn., w/ order} \]

\[ H_m^{(0)}(z) \text{ is a soln. w/ certain}\]

\[ \text{singularity at } z=0 \text{ & outgoing.} \]

\[ \text{eq.} \quad H_0^{(0)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\pi}{4})} \left[ 1 + O\left(\frac{1}{z}\right) \right], \quad z \to \infty \quad \text{asymptotic} \]

\[ \text{decaying complex exponential.} \]

\[ \text{fact: } H_0^{(0)}(kr), \text{ hence } \mathcal{F}(x,y) \text{ valid for } x,y \in \mathbb{R}^3, \text{ sat. radiation condition} \]