NSF DAY AT DARTMOUTH
September 11, 2008

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\[ -\Delta u - \xi u = 0 \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{on} \quad \partial \Omega. \]

- Pick a freq. \( \xi \).
- Set up \( u = \sum \alpha_n \phi_n(\xi) \) basis forms.

\[ \text{boundary norm} \quad \| u \|_{L^2(\partial \Omega)}^2 = \sum \alpha_n \| \phi_n(\xi) \|^2_{L^2(\partial \Omega)} \]
\[ = \sum_{nm} \alpha_n \alpha_m \int_{\partial \Omega} \phi_n(\xi) \overline{\phi_m(\xi)} \, ds \]

- \( \| u \|_{L^2(\Omega)}^2 = \| u \|_{L^2(\partial \Omega)}^2 \) interior norm.
- If \( u \) sat. \( (\Delta + \xi)u = 0 \) in \( \Omega \),

\[ t[u] = \frac{1}{\| u \|_{L^2(\Omega)}} \quad \text{measures how close} \quad u \quad \text{is to eigenfunc of laplacian}. \]

Why? \( t[u] = 0 \Leftrightarrow u \quad \text{is eigenfunc}. \]

Method: i) guess \( E \)
ii) Find \( \min \{ t[u] : u \in \text{Span} \{ \phi_n(\xi) \} \} \)
iii) search along \( E \) axis for minimum where \( t[u] = 0 \).

\[ t_\text{min}(E) = \min \{ \frac{\xi}{\alpha_n} : \alpha_n \geq 1 \} \]

Rayleigh quotient for matrix \( (F,E) \)

\[ t_\text{min}(E) = \frac{\| \phi \|^2}{\| \psi \|^2} \]

Let \( G \) be positive definite Hermitian, \( F \) Hermitian, both N.M. generalized.

Min. value of Rayleigh quot. \( (F,G) \) is the minimum eigenval \( \lambda_1 \) of \( Fv = \lambda Gv \).

\[ \lambda_1 = \min \{ \frac{\| \phi \|^2}{\| \psi \|^2} : \phi \in \phi_n \} \]

and is achieved at \( \xi = \lambda_1 \) the correct degree.

**Lemma:** spectral thm. for pair \( (F,G) \): \( G \) has complete set \( \{ \psi_n \} \) eigenvecs which sat. \( \psi_n^* G \psi_n = \sum \lambda_i \psi_n \), \( \psi_i \) orthogonal.

*Proof:* spectral thm. for \( G \):

\[ G = \Lambda W^* W \Lambda \Sigma \] \( \Lambda = \text{diagonal, } \Sigma = \text{positive definite} \)

coordinate change \( x = \Lambda^{1/2} W^* y \)

\( y \in \text{cans.} \)

So \( Fv = \lambda Gv \Leftrightarrow FW^{1/2} \Lambda^{1/2} x = \lambda v \Leftrightarrow \Lambda^{1/2} W^* F W^{1/2} \Lambda^{1/2} x = \lambda x \)

\( L \text{ is Hermitian since } F \text{ is}. \)
By spectral then \( A^{1/4} W F W A^{-1/4} = X \Lambda X^* \) for some \( \Lambda \).

Then \( V_i = W A^{1/4} x_i \), \( i = 1, ..., N \) are eigenvalues of \( G E P \), basis for \( \mathbb{C}^N \).

\[ V_i \Lambda W^{1/2} G W A^{1/2} x_j = \delta_{ij} \]

QED.

New pf thm: any \( \alpha \) can be written \( \sum_{i=1}^N \beta_i V_i \) since \( V_i \) basis.

\[ F \alpha = \sum_{i=1}^N \beta_i \mu_i \text{G} V_i \]

\[ \alpha^* F \alpha = \sum_{i,j=1}^N \beta_i \bar{\beta}_j \mu_i \text{G} V_i \text{G} V_j \delta_{ij} = \sum_{i=1}^N \mu_i |\beta_i|^2 \]

\[ \alpha^* G \alpha = \sum_{i=1}^N \bar{\beta}_i \beta_i \text{G} V_i \text{G} V_i = \sum_{i=1}^N |\beta_i|^2 \]

so \( \frac{\alpha^* F \alpha}{\alpha^* G \alpha} = \frac{\sum_{i=1}^N |\mu_i|^2}{\sum_{i=1}^N |\beta_i|^2} \) minimized by choosing \( \beta_i = 0 \) for \( i \neq j \).

G's ellipsoid becomes sphere.

ES aligned w/ axes.

May also work via Lagrange multipliers.

In practice, we find as \( N \) incr., \( \phi_n \) was close to \( \phi_n \).

\( \Rightarrow \) F, G acquire common numerical nuspace, not full rank. (in floating pt.)

Recipe (based on Fix & Krebslger, 1972, Vargi-Sornaraj 1991):

\[ \text{Step 1: } G \rightarrow W, \quad \Lambda \]

\[ \text{Step 2: } \tilde{W} \leftarrow W F W A^{-1/2} \rightarrow X, \quad D \]

\[ \begin{pmatrix} W \end{pmatrix}_D \]

How fill \( F, G \) correctly? Briefly each needs \( O(N^2) \) integrals, each w/ quadrature.

\[ F_{mn} \approx \sum_{j=1}^N w_j \phi_m(y_j) \phi_n(y_j) = \sum_{j} A_{mj} A_{jn} \]

where \( A_{jm} = \phi_m(y_j) \)

\[ \text{Step 1: } G = A^* A \]

\( \text{Step 2: } B_{jn} = \sqrt{w_j} \phi_j(z_j) \]

when \( I_{D} = 1 \).

Note \( E_{min} = \min \sum_{j=1}^M E_{j} \)

\( \text{Betchis's paper uses this ...}

\text{Error analysis: if } E_{(j)} \text{ small, how close is } E_{j} \text{ to signal } E_{(j)} \text{ of domain?}

\text{How? (Or else) } u = 0 \text{ in } A,

\[ \text{hence } \left| E_{j} - E_{j} \right| \leq G_{ij} \text{ can be done only on domain } \]

\[ \text{relative error in eigenvalue } \mu \text{ means...} \]

\[ \sqrt{\mu N} \approx \text{Eigenv to \( E_{j} \).} \]
State Thm. on WS. give example of domain: \( \Delta \) in \( \mathbb{R}^2 \) may act on faces in \( \mathbb{C}^2 \).

* self-adjointness of \( \Delta \) in \( L^2 \) only w/ appropriate BCs (p. 18).

\( A = -\Delta \) will \( D(A) = H^s \cap \mathcal{C}^1 \), \( u|_{\partial\Omega} = 0 \).

Let \( u \in D(A) \) and \( w \) be st. \( u-w \in D(A) \) \( \forall w \in \mathbb{C} \), ie \( w \) has same body values as \( u \).

\[ u-w = E u \] \( \forall u \in \mathcal{C} \). \( E \) \( \subset \mathcal{C} \).

For Thm. \( E \) is st. \( u \) is st. in \( \partial \Omega \), but not neces. \( u \) is st. body.

\[ \Delta \text{ extension of operator } A \text{ to larger domain, i.e. } D(A) \subset D(A) \subset H \]

prove Thm. Let \( u \in D(A) \) and \( w \) be st. \( u-w \in D(A) \) \( \forall w \in \mathbb{C} \), ie \( w \) has same body value as \( u \).

\[ E u \]

\[ (\Delta - E)u = 0 \]

and \( E \geq 0 \). \( (\Delta - E) \geq 0 \).

\[ \sum c_i (E-E_i)^2 \leq \sum c_i (E-E_i) \phi_i \leq E \| w \|^2 \]

since \( E(u-w) - \Delta(u-w) = (\Delta - E)w = Ew \)

\[ \sum \frac{c_i (E-E_i)^2}{E_i} \leq \min_i \left( \frac{E-E_i}{E_i} \right)^2 \cdot \sum c_i \frac{E_i}{E_i} \leq E \| w \|^2 \]

so \( \min_i \left( \frac{E-E_i}{E_i} \right) \leq \frac{\| w \|^2}{\| w \|^2} \)

A-posteriori estimates.

Finally \( \| w \|^2 \leq C \left( \frac{\| w \|^2}{\| w \|^2} \right) \)

(Similar bound on eigenvalue.)

Use for error bounds on disk to eigenvalue.
Error and of MPS for eigenmodes: $\phi_j$ be eigen $E_j$ of domain $\Omega$, then (we'll show)

Let $(A+E)u = 0 \quad u \in \Omega$,
then $E_j$ st. \[ \frac{\| E - E_j \|}{\| E_j \|} \leq c_2 \epsilon j \] (cons.) only on $\Omega$ in $\epsilon_j \sim O(1)$

Usage: \[ \frac{\| E - E_j \|}{\| E_j \|} \] rel. err. in $E_j$, then smaller $\epsilon_j$ is, smaller closer that must be $E_j$ true eigen.

\[ \epsilon_j \sim 10^{-16} \] (sane)

\[ \epsilon_j \sim 10^{-16} \] rel. err.

Domain of $op. A = D(A) \subseteq \mathcal{H}$ Hilbert space, complete inner product 2-norm

\[ \mathcal{H} = C^0(\Omega) \quad A = -\Delta \quad D(A) = \{ f \in C^0(\Omega) \mid f \mid_{\partial \Omega} = 0 \} \] is also cont. on $\partial \Omega$.

Note $D(A) \neq \mathcal{H}$ since $A$ not bounded, so can't act on every element of $\mathcal{H}$ (since $\mathcal{H}$ complete, contains its limit points it is closed).

We assume $A$ has point spectrum, ie. countable set of eigenvalues $E_j$, whose eigenvectors are complete and form a dense subset.

Then: define $A^*_{opt}$, true for diff. ops since kernel of $A^*$ is Greens function, continuous or weakly singular.

\[ \text{Then: } \text{define } A^*_{opt} = A u - E u, \quad \text{for } u \in D(A). \]

\[ \text{then: } J_{E_j} \text{ st. } \frac{\| E - E_j \|}{\| E_j \|} \leq c_2 \epsilon \]

WS:

Similarly we prove: \[ \exists \text{ eigenmode } \phi_j \text{ st. } \frac{\| u - \phi_j \|}{\| \phi_j \|} \leq c_2 \epsilon \] where $\epsilon : = \frac{\| E \|}{\| A \|}$ $A_j(x)$.

Application: for above $A$, $D(A)$, say $E_n$ produced by some method (eg. FEM)

\[ E_n(x) \approx \min \left\{ E \mid E_k \neq E_j \right\} \quad \text{dist to next nearest equal}. \]

From Müller-Payer's:

\[ \text{choose } A, D(A) \text{ as above, and an extension of } A \text{ to larger domain } D(A) = D(A) \subseteq \mathcal{H}. \]

This means $\hat{A} |_{D(A)} = A$, ie $\hat{A} u = A u \quad \forall u \in D(A)$. 

\[ \text{choose } D(\hat{A}) = C^0(\Omega) \cap C(\overline{\Omega}) \quad \text{with no BCs on } \Omega. \]
Approximation theory for MPS for Laplace BVP: 

$\Omega$ bounded, $\psi$: analytic, simply-connected, in $\mathbb{R}^2$. 

Thus: $\phi_\alpha$ are dense in the space of solutions to $\Delta u = 0$, i.e., for any soln. $\Delta u = 0$ in $\Omega$, 

$$ \lim_{N \to \infty} \inf_{\phi \in \mathcal{A}_N} \| u - \phi \|_{L^2(\Omega)} = 0 $$

as $N \to \infty$. 

Proofs: Gauß bds.

- Same result in sup norm.
- Example of Runge's theorem (approximation of any analytic function on $\Omega$ by polynomials of degree $p$).

Re-expression in special case: $\Omega = \mathbb{C}$, only polynomials of degree $\leq p$ needed.

- Connection to analytic functions: every harmonic function can be written as $\Re \{ f \}$ for $f$ analytic.

- If $\Omega$ is simply connected, need basic functions to include singularities in each connected component of $\mathbb{R}^2 \setminus \overline{\Omega}$.

If soln. $u$ may be analytically continued beyond $\Omega$ in a finite distance, 

and if $u$ is analytic on $\Omega$, $u$ is analytic in $\overline{\Omega}$.

Then 

$$ \lim_{N \to \infty} \inf_{\phi \in \mathcal{A}_N} \| u - \phi \|_{L^2(\Omega)} = 0 \quad \text{for every } \Omega \subset \mathbb{R}^2 $$

Furthermore, $\Omega$ is connected.

Def: conf. dist. $p(x)$ for $p \in \mathbb{R}^2 \setminus \overline{\Omega}$: solve

$$ \Delta v = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega} $$

$$ v = 0 \quad \text{on } \partial \Omega $$

Then $p(x) = e^{v(x)}$.

- Note how similar to quadrature convergence rates.

Through theory of Velinov (see Friedrichs thesis, Henri's review), convergence rate carry over to 2nd-order elliptic PDE w/ analytic coefficients. eg. $(A + \varepsilon)u = 0$ Helmboldt.

EVP behavior at corners:

- $\beta = \frac{\pi}{n} \in \mathbb{Z}$ called regular: mode $\phi_j$ analytically continued beyond corner, by sin-fold reflection.

- Otherwise singular, eg.

- Note HW7

- $\frac{\pi}{3}$ all regular, = eigenmode, analytically continued, in fact to $\mathbb{R}^2$, since $\Omega$ is plane

- Singular corner:

- To regain exponential convergence near $J_\nu(kr) \sin \theta$ for $\nu = \frac{\pi}{3} n$ necessary.

show mushroom pets?