We want unitary $Q_1$ turning $a_1 \in \mathbb{R}^n$ into $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = Q_1 a_1$ (x)

a) How big is $a_1$?

Now find the matrix $H$ which reflects $y \in \mathbb{R}^m$ through the plane through the origin with normal $v \in \mathbb{R}^m$:

Hints: use picture, try $\|y\|=1$ case first.

b) You should get $H = I + (\text{rank-1})$. What minimum effort is needed to apply this to a general vector $y$, i.e. the matrix $x = Hy$?

We pick $Q_1$ of the form $H$:

c) Explain why if (x) holds then $v$ must be of the form $v = (a_1 + [v])$ & we can choose $x = 1$.

d) Let $f = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$, i.e. $a_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ and so $v = \begin{bmatrix} a_1 + f \\ \vdots \\ f \end{bmatrix}$. By considering (x) & your formula for $H$, solve for $f$ & simplify:

**BONUS:** which of the two solutions has less relative error due to rounding error?
We want unitary $Q_1$ turning $a \in \mathbb{R}^n$ into $\begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Q_1a$.

(a) How big is $\alpha$? $\alpha = \|a\|$ since $\|Q_1a\| = \|a\|$ by unitarity.

Now find the matrix $H$ which reflects $y \in \mathbb{R}^m$ through the plane through the origin with normal $v \in \mathbb{R}^m$:

$$x = y - 2\text{ (this vector)} = y - 2v(v^Tv)$$

$$H = I - 2\frac{vv^T}{v^Tv}$$

Hints: case picture. Try $\|v\| = 1$ case first. This vector = $(v^Tv)v$ in the case $\|v\| = 1$.

(b) You should get $H = I - (\text{rank-1})$. What minimum effort is needed to apply this to a general vector $y$, i.e. the matrix $x = Hy$?

We pick $Q_1$ of the form $H$:

$$Hv = y - \frac{2}{v^Tv}v(v^Tv)v$$

$O(m)$ effort. $O(n)$ substitution.

(c) Explain why if $(n)_v$ holds then $v$ must be of the form $v \left( a_1, \frac{0}{0} \right)$ and we can choose $\nu = 1$. Since $H_{a_1}$ must be $\left[ \frac{0}{0} \right]$ then $\nu \left[ \frac{0}{0} \right]$ must be $\alpha \left[ \frac{a_1}{a_1} \right]$.

(d) Let $f = \left[ \begin{array}{c} \frac{a_1}{a_1} \\ \frac{a_2}{a_2} \\ \vdots \\ \frac{a_n}{a_n} \end{array} \right]$ i.e. $a_i = \left[ \begin{array}{c} a_i \\ f \end{array} \right]$ and so $v = \left[ \begin{array}{c} a_i + \beta \\ f \end{array} \right]$.

By considering $(x)$ & your formula for $H$, solve for $\beta$ & simplify:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = H_{a_1} = a_1 - 2\frac{v_a}{v^Tv}v = \left[ \begin{array}{c} a_1 \\ f \end{array} \right] - 2\frac{v_a}{v^Tv} \left[ \begin{array}{c} a_i + \beta \\ f \end{array} \right]$$

Since $f$'s cancel, we must have $2\frac{v_a}{v^Tv} = 1$.

So, $v^Tv = 2v_a$, i.e. $a_i^2 + 2\beta a_i + \beta^2 + \|f\|^2 = 2(a_i^2 - \beta a_i + \|f\|^2)$ i.e. $\beta = a_i^2 + \|f\|^2 - \frac{\beta}{2\|f\|^2}$. $eta \approx \frac{\|f\|^2}{2}$.

Using: which of the two solutions has less relative error due to rounding error? $\text{sgn}(a)$.