\[ H = I - \alpha \frac{vv^T}{\|v\|^2} \]

\[ v = a_1 - \epsilon, \|a_1\| \]

\[ \begin{pmatrix} a_1 \\
0
\end{pmatrix} \]

\[ \begin{pmatrix} 1 \\
0
\end{pmatrix} \]

\[ \text{two choices of plane } v. \]

\[ \text{pick sign?} \]

\[ \text{weight } \sqrt{\|a_1\|} = \begin{pmatrix} a_1 \\
\epsilon
\end{pmatrix} \begin{pmatrix} a_1^T \\
\epsilon^T
\end{pmatrix} \]

\[ = a_1^2 + \|a_1\|^2 + \|\epsilon\|^2 \]

\[ \text{let small, as in fig.} \]

\[ \text{want add, twist cancel (C.C.)} \]

\[ \text{so } v = a_1 + \text{arg}(a_1) \|a_1\| e_1 \rightarrow \text{inverse which in fig?} \text{long one.} \]

\[ \text{remaining cols?} \]

\[ Q_1 A = \begin{bmatrix}
R_1 & 0 \\
0 & 0
\end{bmatrix} \]

\[ \text{now work in } R^{m-1} \text{ & find } \tilde{Q}_2 \in R^{(m-1)\times(m-1)} \]

\[ \text{so } x \rightarrow \tilde{x} \]

\[ \text{let } \tilde{Q}_2 = \begin{bmatrix} 1 & 0 \\
0 & \tilde{Q}_2
\end{bmatrix} \]

\[ \text{why is mess up by } \tilde{Q}_2? \text{ since in } R^{m\times 1}, \text{ all } 0. \]

\[ \text{finally } Q_m \cdots Q_1 A = R = \begin{bmatrix} \tilde{Q}_1 \tilde{Q}_2 \end{bmatrix} \]

\[ \text{triangulization by orthog steps: } \]

\[ Q : R = \tilde{Q}(A) \rightarrow \tilde{Q} \text{ unitary} \]

\[ \text{How get } Q \text{ st. } A = QR? \]

\[ Q = Q_1^{-1} \cdots Q_m^{-1} \rightarrow \text{cost? } O(m^3) \text{ apply, } n \text{ times each step} \]

\[ \text{show: } \tilde{Q}_2 (x) = \tilde{Q}_1 (\tilde{x}) \]

\[ \text{how it is done.} \]

\[ \text{can prove B.S. : } \tilde{Q}_2 = A \times \delta A \text{ for } \delta A = O(\text{small}) \]

\[ 40\text{min}. \]
FFT

Cooley-Tukey paper. 1965 credits — Good ‘58

but actually Fourier series in 1820s a UK, for analyzing tides. A Ruzewski, 1906.

Cauchy 1805 trivariate, for orbit of asteroids.

drawn by lexicographical application.

Idea of FFT in exam mutation, (not paper):

\[ f \in \mathbb{C}^N \]

\[ \hat{f} = Ff \]

\[ F_{mn} = e^{2\pi i mn/N} \]

Defn: \[ x = e^{2\pi i n/N} \]

\[ F_{mn} \in \mathbb{C}^{N \times N} \]

F's in \[ \mathbb{C}^{N \times N} \] elements: \[ F_{mn} = w^{-mn} \]

Discrete FT: \[ \hat{f} = Ff \]

ie \[ \hat{f}_m = \sum_{j=0}^{N-1} w^{-mj} f_j \]

for \( m = 0, \ldots, N-1 \) \( \mathbb{C} \)-indexed

\[ F \] is the paradigm ‘Fast algorithm’ = connect to FMM (2x6), Rader-Sab (2x6).

\[ \text{Inverse Complexity? } O(N^2) \text{ worse.} \]

Sum Lemma: \[ \sum_{k=0}^{N-1} w^{mk} = \begin{cases} N & j \equiv 0 \pmod{N} \\ 0 & \text{otherwise.} \end{cases} \]

\[ \|Ff\|_2 = \|f\|_2 \]

Claim: If even. DFT formula outside \( 0 \leq m < N \), the \( N \)-periodic:

\[ \tilde{f}_{m+kN} = \frac{1}{N} \sum_{j=0}^{N-1} w^{-m(j+kN)} f_j \]

Say \( N \) even: \[ \tilde{f}_{m+kN} = \frac{1}{N} \sum_{j=0}^{N-1} w^{-m(j+kN)} f_j = \frac{1}{N} \sum_{j=0}^{N-1} w^{-mk} f_{j+kN} + \sum_{k=0}^{N-1} w^{-mk} f_{2kN} \]

\[ \tilde{f}_m = \sum_{k=0}^{N-1} \frac{1}{N} \sum_{j=0}^{N-1} w^{-mk} f_{j+kN} = \frac{1}{N} \sum_{j=0}^{N-1} f_j \]

What \( w \) would appear as length \( \sqrt{N} \) DFT? \( e^{2\pi i/N} = w \).
So, if $0 \leq m < N/2$,

$$
\delta_m = (F^{(m)}f)^m + \omega^{-m} (F^{(-m)}f)^m.
$$

2nd half of answer?

$$
\sum_{m=N/2}^{N-1} \delta_m = \sum_{m=N/2}^{N-1} \omega^{-m} \delta_m
$$

$$
= C_m - \omega^{-m} D_m,
$$

where $C_m = F^{(m)}f$.

Danielsen - Lanczos Lemma.

Algorithm for length-$N$ DFT of $f$:

1. Shuffle $f$ into $f^0, f^e$.
2. Set $\tilde{f} = F^{(m)}f^e$.
3. Set $\omega = F^{(m)}f^0$.
4. Stack $\hat{f} = [\tilde{f} + W\tilde{\omega}] W = \text{diag} \{\omega, ..., \omega^{N/2}\}.$

Assume $N = 2^m$; how apply above? recursion.

Stop when $f^e, f^0$ length 1:

$$
\delta = \tilde{f} = f^e,
\quad \tilde{\delta} = f^0.
$$

Total: $O(N \log N)$, for $N=10^6$, how much time?

~ $10^5$ ms, huge.

Applications:

- Convolution of signals (blurring, deblurring)
- Fast multiplication of integers (number theory)
- Solving PDEs.
- Extracting components of signals.
- Processing, compression (MP3, JPEG).

5 min feedback Q's.