\[ \Delta u = 0 \quad \Rightarrow \quad u = \text{harmonic function} \]

\[ \Phi(x, y) = -\frac{1}{2\pi} \ln |x - y| \]

\(d=2\quad x, y \in \mathbb{R}^2\quad \text{or} \quad \mathbb{C}^2 \quad \text{let} \quad \mathbf{r} = x - y \quad \text{vector} \quad \mathbf{r} \]

\(d=3\quad x, y \in \mathbb{R}^3 \quad \text{I}\)

\(\text{free-space Green's function : as}\) \(f\) \(\text{fun} \quad \text{of} \quad x \quad \text{the potential due to charge at} \quad y \quad \text{symm.} \quad k \rightarrow 0 \quad \text{symm.} \)

Then: \(\Phi(x, y)\) \(\text{harmonic in} \quad \mathbb{R}^d \setminus \{ \mathbf{y}\}\)

\(\text{the set of all points minus the single point} \quad \mathbf{y} \quad \text{by} \)

\(\text{pf: without loss of generality,} \quad y = 0\)

\[ \text{eg. } d=2 \quad \frac{\partial^2}{\partial x_1^2} \ln |x| = \frac{1}{|x|} \ln (x_1^2 + x_2^2) = \frac{1}{2} \cdot 2x_1 \frac{1}{x_1^2 + x_2^2} = \frac{x_1}{|x|^2} \]

\[ \frac{\partial^2}{\partial x_1^2} \ln |x| = \frac{1}{|x|^2} + \frac{2x_1}{|x|^2} \frac{1}{x_1^2 + x_2^2} = -\frac{2x_1^2}{|x|^4} \]

\[ \Delta \ln |x| = \frac{1}{|x|^2} - \frac{2x_1^2}{|x|^4} + \frac{1}{|x|^2} - \frac{2x_1^2}{|x|^4} = 0 \]

In classical notion of finite A deriving \(\Delta (\Phi(x, y))\) doesn't work at \(x = y\).

What can broaden over reach to include 'smooth distributions' (Debunked in M 6.2)

\(\text{let} \quad f(x) \quad \text{is} \quad C^n \quad \text{smooth, finite, vanishing outside some bounded region} \quad \text{then} \quad \delta(x) \quad \text{is any linear functional of} \quad f \quad \text{e.g.} \quad f(0) \)

\(\text{Dirac delta} \quad \delta(x) \quad \text{is a distribution} : \quad \int f(x) \delta(x - x_0) \, dx = f(x_0) \)

Distributions is no function \(\delta(x)\) but use as abbreviation.

Note: for operator \(L = -\Delta\)

\(\text{it turns out} \quad L \Phi(x, y) = \delta(x - y) \quad \text{in sense of distrib}, \)

\(\Phi\) is kind of inverse of \(L\)
\[ \mathcal{L} = \text{domain with sufficiently smooth boundary } \partial \mathcal{L}, \text{ } u, v \text{ sufficiently smooth functions} \]

Green's Theorem:

\[ \iint_{\Omega} \left( u \Delta v + \nabla u \cdot \nabla v \right) \, dx = \oint_{\partial \Omega} u \, v \nu \, ds \]

\[ \iint_{\Omega} \left( u \Delta v - v \Delta u \right) \, dx = \int_{\partial \Omega} \left( u v_n - v u_n \right) \, ds \]

**Proof:**

\[ \nabla \cdot (u \nabla v) = u \Delta v + \nabla u \cdot \nabla v \]

\[ \iint_{\Omega} dx \text{ both sides of } \text{ use } \iint_{\Omega} \nabla \cdot (u \nabla v) \, dx = \int_{\partial \Omega} \nabla \cdot (u \nabla v) \, ds \]

**GT2:** subtract GT4 with \( uv \) from GT1.

How smooth? \( \Omega \): To prove things (analysis of PDEs) mathematicians have litany of classes for domains eg. \( \Omega \) is \( C^k \)

\( C^k \) means \( x(t) \), where \( t \) parameterizes boundary is a \( C^k \) function, is \( k \) derivatives up to order \( k \) are continuous.

eg. \( \Omega \) is 'precise' \( C^\infty \)

Lipschitz: \( \nabla \) locally graph of \( \partial \Omega \) can be jagged, but derivatives \( \nabla \) \text{ Cont. H"{o}lder continuous}.

Doug Green: This works with corollaries (Kellogg book)

Later we will restrict to \( C^2 \) domains for integral equations to be nice.

\[ \text{Formally } \Omega \text{ is an open set } (\partial \Omega \text{ not included}); \quad \Omega := \mathcal{L} \cup \partial \mathcal{L} \]

\( C^0(\Omega) \): continuous in \( \Omega \) but as approach \( \partial \mathcal{L} \), may blow up \( \infty \).

\( C(\partial \mathcal{L}) \): bounded values also continuous, and are limit of interior as \( \partial \mathcal{L} \).

eg. \( u \in C(\Omega), \forall u \in C^2(\Omega) \text{ guarantees always in } \iint \partial \mathcal{L} \text{ exist classically.}

\textbf{Corollary:} If } u \text{ harmonic, then } \int_{\partial \mathcal{L}} u \nu \, ds = 0.

(Zero flux)
Green's Representation Formula: \( (G.R.F) \)

\[ u(y) = \int_{\partial \Omega} \left[ u(y) \frac{\partial \Phi(x,y)}{\partial y} - u(x) \frac{\partial \Phi(x,y)}{\partial x} \right] ds_y \]

**For \( \kappa \in \Omega \), define \( \partial B(x,r) = \text{circle of radius } r > 0 \) about \( x \)**

For \( y \in \Omega \), \( \Phi(x,y) \) harmonic in \( \{ y \in \Omega : |y-x| > r \} \)

Apply G.T. to region \( \Omega \). 

\[
\int_{\partial \Omega} u(y) \frac{\partial \Phi(x,y)}{\partial y} \, ds_y = -\int_{\partial \Omega} u(x) \frac{\partial \Phi(x,y)}{\partial x} \, ds_y
\]

\[
\Phi(x,y) = -\frac{1}{2\pi} \ln r + \text{constant, } \quad \frac{\partial \Phi(x,y)}{\partial y} = \frac{1}{2\pi} \int_{\partial B(x,r)} \frac{u(y)}{r} \, ds_y
\]

For \( y \in \partial B(x,r) \)

\[
a = -\frac{1}{2\pi} \int_{\partial \Omega} u(y) \, ds_y
\]

*b* = \[\frac{1}{2\pi} \ln r \int_{\partial B(x,r)} u(y) \, ds_y\]

v vanishes by zero flux core = 0

\[
(G.R.F) \quad u(x) = \int_{\partial \Omega} u(y) \Phi(x,y) - u(x) \frac{\partial \Phi(x,y)}{\partial y} \, ds_y
\]

Interior values expressed as boundary integral.

Looking ahead to \( (S\delta)(x) = \int_{\partial \Omega} \delta(y) \Phi(x,y) \, ds_y \)

Single layer potential.

Looking ahead to \( (D\tau)(x) = \int_{\partial \Omega} \tau(y) \frac{\partial \Phi(x,y)}{\partial y} \, ds_y \)

Doubly.

Then \( G.R.F \) says \( u = S\delta + D\tau \)

\[
\text{With densities given by boundary values of } u : \tau = \left. u\right|_{\partial \Omega} \quad \delta = \left. u\right|_{\partial \Omega}
\]

Very useful, eg: Poiseuille...
Mean Value Theorem for harmonic functions:

\[ \text{avg. of a harmonic func over any sphere } \quad \text{at center} \]

Proof: Let \( \Omega \) be open ball \( \{y \in \mathbb{R}^d : |y-x| < R \} \) in \( \mathbb{R}^d \). Suppose \( u \in C^0(\bar{\Omega}) \), and for any \( x \in \Omega \),

\[ u(x) = \frac{1}{2\pi} \int_{|y-x|=R} u(y) \frac{1}{|x-y|} \, ds_y. \]

This integral vanishes by vanishing of harmonic functions at infinity.

We could integrate this over \( 0 < r < R \) to get the average over the whole ball.

\[ \frac{1}{4\pi} \int_{|y-x|=R} u(y) \, ds_y = \text{avg. on sphere } = \int_{|y-x|=R} u(y) \, ds_y \]

for \( d=3 \), its surface area.

**Maximum Principle:**

Max & min of harmonic functions must occur on \( \partial \Omega \), unless \( \Omega \) is a ball

Proof: Suppose there is a max. at some \( x \in \Omega \). Then there is some sphere around \( x \) within \( \Omega \),

No value on sphere can exceed \( u(x) \).

\( \Rightarrow \) By MVT, all values must equal \( u(x) \).

True for all radii less than \( R \).

Can now repeat using \( x \) another point in \( \Omega \).

Repeat this process in \( \Omega \).

Repeat for min values.

**Uniqueness of interior Dirichlet BVP:**

Find \( u \) harmonic in \( \Omega \) with \( u \mid_{\partial \Omega} = f \).

This has at most 1 solution;

Suppose \( u, v \) were solutions, then \( u-v = 0 \) on \( \partial \Omega \), by Max Principle must vanish in \( \Omega \).

\[ \Rightarrow u = v. \]

**Note:** haven't proved existence. This is done via potential theory (coming up).

Here \( u \) is a 'classical solution' i.e. \( u \in C^2(\Omega) \cap C(\bar{\Omega}) \) with \( f \in C(\partial \Omega) \).

There are also 'weak' solutions, when doing min-max principles. The above is reformulated for \( \Delta u \in C^2(\Omega) \).
Last time we used Maximum Principle for harmonic forms to prove uniqueness for interior Dirichlet BVP, for classical solutions in domains for which Dirichlet's Theorem holds.

**Remarks:**

1) the exterior Dirichlet BVP in $d=3$ also can be proved unique this way:

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega} \\
\frac{\partial u}{\partial n} &= f \\
\end{aligned}
\]

\[u(x) = o(1) \quad \text{as } |x| \to \infty, \quad \text{uniformly on any angle} \frac{1}{|x|}.
\]

"Little oh", i.e. vanishes. The problem is only unique "without the condition."

Apply the Max. Princ. to $B(R) \setminus \bar{\Omega}$, and take $R \to \infty$.

The difference of two solutions $u_1, u_2$ satisfies $\Delta u = 0$ and

\[\max_{x \in \partial B(R)} |u_1 - u_2| \quad \text{smaller than any given constant as } R \to \infty \Rightarrow u = 0\text{ in } B(R) \setminus \bar{\Omega}.
\]

2) We have not yet proven existence of classical solutions; one way is via integral operators (coming up).

3) Verdik, Kenig (see Kenig 1994 CBMS regional conference notes §23).

have proven uniqueness & existence even for Lipschitz bounded domains.

This involves the idea of harmonic measure & is quite advanced (I don't know it).

- Monge-Ampere
- H1-Nehari
- etc.

Double layer is just setting $\varepsilon = \gamma y$ with $y \in 2\alpha$, integrating along bounding.
POTENTIAL THEORY

\[ \begin{aligned}
\text{Single layer } (S\sigma)(x) &= \int_{\partial\Omega} \sigma(y) \delta_g(y) \, ds_g, \quad \sigma \in C(\partial\Omega) \quad \text{and} \\
\text{Double layer } (D\tau)(x) &= \int_{\partial\Omega} \tau(y) \frac{\partial}{\partial n} \delta_g(y) \, ds_g, \quad \tau \in C(\partial\Omega)
\end{aligned} \]

are both harmonic for \( x \notin \partial\Omega \) (proof: integral continuous, differentiable under integral sign).

What happens as \( x \to \partial\Omega \)? Sometimes depends which side you're on!

For \( x \in \partial\Omega \), define \( U^\pm(x) := \lim_{h \to 0^+} U(x \pm h \hat{n}_x) \)

\[ U^\pm_n(x) := \lim_{h \to 0^+} \hat{n}_x \cdot \nabla U(x \pm h \hat{n}_x) \]

- Thm (Jump Relations) Let \( \Omega \) be class \( C^2 \), \( \tau, \sigma \in C(\partial\Omega) \), \( \nabla \cdot \sigma = 0 \)

\( u \) continuous everywhere in \( \mathbb{R}^d \), i.e. \( U(x) = \int_{\partial\Omega} \Phi(x,y) \delta_g(y) \, ds_g \) on \( x \in \partial\Omega \)

\( u^\pm_n(x) = \int_{\partial\Omega} \frac{\partial \Phi(x,y)}{\partial n_x} \delta(y) \, ds_g \pm \frac{1}{2} \delta(y), \quad x \in \partial\Omega \)

(\* note \( x \neq y \)!

\( \tau \) normal derivative. \( \tau^+ = \tau^- \)

\( V^\pm(x) = \int_{\partial\Omega} \frac{\partial \Phi(x,y)}{\partial n_y} \tau(y) \, ds_g, \quad x \in \partial\Omega, \) i.e. normal derivative. \( V^+ = V^- \)

\( V^\pm(x) = \int_{\partial\Omega} \frac{\partial \Phi(x,y)}{\partial n_x} \tau(y) \, ds_g \pm \frac{1}{2} \tau(y), \quad x \in \partial\Omega \)

\( \) jump!

The above integrals are improper (since \( x, y \) both on \( \partial\Omega \), integral undefined for \( x = y \))

but singularities, if present, are integrable.

E. g. \( 1/|x-y| \) singularity in \( \mathbb{R}^2 \), integrable along \( \{ \text{line} \} \); (even if \( d \leq 3 \), harmonic) \( \) 2-surface
Proofs are hard (Milkia, Cotton-Kron books — we may get to?)

Why the jumps? Heuristically...

\[ \Phi(x,y) = \int_{A_r} \Phi(x,y) \delta(y) \, dy + \int_{\Gamma_r} \Phi(x,y) \delta(y) \, ds \]

well behaved as \( r \to 0 \), gives integral in i).

as \( s \to 0 \), \( \frac{1}{r} \int_{\Gamma_r} \ln|s| \, ds = 0 \)

& \( \Gamma_r \) is bounded \( \Rightarrow \) vanishes,

\( s \) is arclength.

But \( \Phi(x,Y) = \lim_{n \to 0} \int_{A_r} \frac{\partial \Phi(x+h_n, y)}{\partial n_x} \, dy + \lim_{n \to 0} \int_{\Gamma_r} \frac{\partial \Phi(x+h_n, y)}{\partial n_x} \delta(y) \, ds \)

Take \( e \to 0 \) but \( \frac{h}{e} \to 0 \), i.e. \( h \ll r \)

\( \Phi \) is integral gives integral in ii)

approx flat, use \( \frac{\partial \Phi(x+h_n, y)}{\partial n_x} \approx \frac{\cos \theta}{2\pi \sqrt{m^2 + s^2}} \)

\( d \approx \frac{h}{2\pi (m^2 + s^2)} \)

\( \lim_{h \to 0} \frac{1}{2\pi} \int_{\Gamma_r} \frac{b}{m^2 + s^2} \delta(s) \, ds = \frac{1}{2} \delta(s = 0) \)

OR, for physicists, Gauss' law gives potential gradient either side of a sheet of charge.

Jump in \( E \) field \( = \nabla \Phi \)

charge density.

Similar argument for dipole charge sheet, gives jump in \( v \), but same \( \nabla \Phi \) either side.
Summary of Jump relations

JR1 \[ u = S \delta \]
JR2 \[ u_x = D T \phi \mp \frac{1}{2} s \]
JR3 \[ \nu_n = T \uparrow \]
JR4 \[ \nu_x = D T \mp \frac{1}{2} x \]

where \( S, D \) are to be thought of as integral operators: \( C(\mathbb{R}) \to C(\mathbb{R}) \)
\( D^T \) is \( D \) with arguments of kernel swapped.
\( T \) is deriv. of double layer op: \( (T f)(x) = \int_{\partial \Omega} \frac{\varphi(y)}{|x-y|} dy \), kernel
\( T \) is more singular than \( D \) \( \Rightarrow \) I won't make any formal statement here.
(you can in Hölder space).

Example: Double layer with \( \gamma = 1 \) gives, constant \( \psi \) inside, regardless of shape of \( \Omega \):
\[
\int_{\partial \Omega} \frac{\varphi(y)}{|x-y|} dy = \begin{cases} 
-\frac{1}{2} & x \in \Omega \\
0 & x \in \mathbb{R}^d \setminus \Omega 
\end{cases}
\]

Pf: Outside, harmonic in \( \mathbb{R}^d \); \( \nabla \varphi = 0 \) \( \Rightarrow \) apply Green's first (why no flux controll at \( \infty \)?)
Inside, use Green's 1st with \( u = -1 \).
on \( \partial \Omega \), use JR4 with either \( u = -1 \) inside or \( u = 0 \) outside.

You will use this to check numerical accuracy of layer potentials in HW1.

Note in \( d=2 \), with \( \partial \Omega \) class \( C^2 \), \( D \) actually has continuous kernel.
... surprise since \( \nabla \varphi(y) \) diverges like \( O \left( \frac{1}{|x-y|} \right) \).
Prove this later.

Generally, singularity of kernel is crucial:
\( d=2 \), \( \partial \Omega \) 1st integral, \( \int |s|^{-\kappa} ds < \infty \) for \( \kappa < 1 \)
A kernel \( K(s,t) \) is weakly singular if \( |K(s,t)| \leq \frac{C}{|s-t|^{1-\kappa}} \) for \( \kappa < 1 \).
$\alpha = 0$ continuous kernel.

Let's solve interior Dirichlet BVP:

use JR4, set \( v = f \), ask what \( c \) is needed?

\[ \text{Fredholm integral equation of 2nd kind} \]

\[ \text{Then } u(x) = (D^2)^{-1} f \text{ is a solution to Dirichlet BVP}. \]

proof is JR4.4.
Key result from last time: construct a solution to interior Dirichlet BVP using potential theory, a double layer potential.

If \( \tau \) is some function on \( \partial \Omega \) solving \( (\mathcal{D} - \frac{1}{2} I) \tau = f \) \( (x) \)

where \( (\mathcal{D} \tau)(x) = \int_{\partial \Omega} \frac{\partial \Phi(x,y)}{\partial n_y} \tau(y) \, dy \) is possibly improper if \( x \in \partial \Omega \)

Then \( \Phi(x) = (\mathcal{D} \tau)(x) \), \( x \in \Omega \) is a solution to \( \Delta u = 0 \) in \( \Omega \)
\[
\begin{aligned}
\quad \text{\( u|_{\partial \Omega} = f \).}
\end{aligned}
\]

This is not just an analytic tool; it will give us an efficient numerical method.

The boundary integral equation (BIE) labeled \( (x) \) is a Fredholm 2nd kind integral equation:

\[
\begin{aligned}
K \tau = f \quad &\text{"1st kind"}\quad \rightarrow \text{nasty to invert for smoothing on } K
\end{aligned}
\]

\[
\begin{aligned}
K \tau - \tau = f \quad &\text{"2nd kind"}\quad \rightarrow \text{well-behaved to invert (solve)}.
\end{aligned}
\]

The 1st kind is nasty since many \( K \) arising in practice are smoothing (and "compact") in which case \( K^{-1} \) does not exist as a bounded operator (\( K \) is not "injective")

How singular is kernel of integral op \( \mathcal{D} \) ?

Recall \( \mathcal{D}(x,y) = \frac{\partial \Phi(x,y)}{\partial n_y} = \frac{1}{\text{vol}(\text{d-dim unit-ball})} \) for \( x \in \partial \Omega \), \( \text{vol} = \text{surf. area of d-dim unit-sphere} \).

So for general \( x \in \Omega \) with convex \( \mathcal{D} \) is strongly singular; \( \mathcal{D}(x,y) = O\left(\frac{1}{|x-y|^{d-1}}\right) \).

But for \( C^2 \) domain, \( x \in \partial \Omega \), can bound \( \mathcal{D}(x,y) \leq L |x-y|^2 \) \( (\text{book Cotton-Kress '83}) \)

\[
\begin{aligned}
\mathcal{D}(x,y) \leq \frac{c}{|x-y|^{d-2}} \quad \text{which is only weakly singular}
\end{aligned}
\]

Recall 'weak' singularity is integrable (on \( \partial \Omega \)) but 'strong' is not: \( \int_\partial \mathcal{D}(x,y) \, dy < \infty \) for \( \alpha < d-1 \)

for \( x \in \partial \Omega \), since \( \partial \Omega \) is of dimension \( d-1 \).
The above suggests \( D(x,y) \) continues for \( C^2 \) domains in \( d=2 \). Let's prove it.

Let \( t \in S^1 = [0, 2\pi) \) parametrize 2D counter-clockwise.
\( x(t) \in \mathbb{R}^2 \) be boundary location.
Unit normal \( n(t) := \frac{(-x'_2(t), x'_1(t))}{|x(t)|} \) vector \( \times \) rotated \( \frac{\pi}{2} \) CCW.

\( C^2 \) means \( \dot{x}(t), \ddot{x}(t) \) cont. (bounded) vector fields.
\( |\dot{x}(t)| > 0 \) \( \forall t \), so 'it always keeps moving',
\( \Rightarrow n(t) \) also cont. vector field.

\[ D(s,t) = \frac{1}{2\pi} \frac{n(t) \cdot (x(s)-x(t))}{|x(s)-x(t)|^2} \]

using dipole formula for \( \frac{\partial D}{\partial n} \) any.

Note since top \& bottom are continuous, \( \dot{x}(t) \) bottom nonzero on \( \{ s,t \in S^1 : s \neq t \} \), so is \( D(s,t) \).

To evaluate \( \lim_{s \to t} D(s,t) \) we recognize top \& bottom both vanish, \( \dot{x} \) so \( \dot{x} \)'s 1st deriv.
\( \Rightarrow \) l'Hôpital rule using 2nd derivs needed.

\[ \frac{\partial}{\partial s} \text{top} = n(t) \cdot \dot{x}(s) \]
\[ \frac{\partial^2}{\partial s^2} \text{top} = n(t) \cdot \ddot{x}(s) \]
\[ \frac{\partial}{\partial s} \text{bottom} = 2 \dot{x}(s) \cdot [x(s)-x(t)] \]
\[ \frac{\partial^2}{\partial s^2} \text{bottom} = 2 |\dot{x}(s)|^2 \]

\[ D(t,t) = \lim_{s \to t} D(s,t) = \frac{1}{2\pi} \frac{n(t) \cdot \ddot{x}(t)}{2 |\dot{x}(t)|^2} \]

exists, \( x \) is continuous w\( \text{r} \)t \( t \).

\[ = -\frac{1}{4\pi R(t)} \]

where \( R(t) = \text{local radius of curvature} \)

\[ = -\frac{x(t)}{4\pi R(t)} \]

\( K(t) = \text{curvature} = \frac{|\dot{x}(t)|}{|x(t)|^2} = \frac{1}{R(t)} \)

So \( D(x,y) \) continues (o bounded) on \( 2\mathbb{A} \times 2\mathbb{A} \).

Thus:

- Let \( G \subseteq \mathbb{R}^m \) be compact set, and \( K : C(\mathbb{G}) \to C(\mathbb{G}) \) the integral operator defined by \( (Kf)(x) = \int_{G} K(x,y) f(y) dy \).
- If \( K(x,y) \) continuous on \( G \times G \) then the operator \( K \) is compact.

So our double layer op. \( D \) is compact for \( C^2 \) domains in \( d=2 \), (eg. Reidel's thm w\( \text{r} \), any functional anal. book).
Recall a set is compact iff every sequence in the set contains a subsequence converging to a point in that set.

For subsets of \( \mathbb{R}^m \) this implies closed \& bounded (with \( m \) is finite).

Say \( X, Y \) are normed spaces, e.g. \( C([0,1]) \) or \( L^2(\Omega) \) etc.

An operator \( K: X \to Y \) is compact iff for each bounded sequence \( \{ x_n \} \) in \( X \), the sequence \( \{ Kx_n \} \) contains a subsequence converging to an element in \( Y \).

Some useful properties of \( K \) compact:

i) \( K \) is bounded operator, i.e. \( \| Kx \| \leq M \| x \| \quad \forall x \in X \).

ii) Spectrum is discrete and eigenvalues tend to zero (Riesz theory).

Recall if \( K: X \to X \), \( \lambda \in \mathbb{C} \) called eigenvalue if \( \exists x \in X, x \neq 0 \) s.t. \( Kx = \lambda x \).

The ‘spectrum’ \( \sigma(K) \) is all points where \( (\lambda I - K)^{-1} \) is not bounded.

Compact ops. have \( \lambda = 0 \) belonging to spectrum, and discrete (countably infinite) set of eigenvalues accumulating only at \( 0 \leq 0 \).

spectral radius = largest \( \lambda \).

iii) Uniqueness \& existence of solution \( u \) to \( K u = f \)

holds if the homog. eqn. \( K u = 0 \) only has trivial solution \( u = 0 \).

In other words \( K \) behaves ‘nicely’ like finite-dim. lin. op. (Riesz-Fredholm theory).

iv) If \( \{ e_n \} \) is orthonormal basis for \( L^2(\Omega) \), then \( \| K e_n \|_2 \to 0 \) as \( n \to \infty \).

(We've specialized to \( K: L^2(\Omega) \to L^2(\Omega) \)). Surprising result! \( K \) is smoothing.

Eg.: \( y_n = \sin nx \) on \( L^2([0,2\pi]) \).

Note that ii) means \( K^{-1} \) is unbounded. \( \to \) bad idea, to invert \( K \) numerically.

We also have: Then: integral operators with weakly singular kernels are compact.

Note that iii) will allow existence of solution to Dirichlet interior BVP to be proved.

---- Beautiful proofs ---- more later...
Numerical approximation of integrals

\[ \int_0^1 f(x) \, dx \approx \sum_{j=1}^{m} w_j f(x_j) \]

weights \quad nodes or quadrature points.

There are many schemes, training for

i) high accuracy, i.e. small error, for certain classes of \( f \);

ii) high-order convergence, i.e. error \( = O(M^{-p}) \), \( p = \) order large.

tj) see Numerical Recipes. Note i) & ii) not always compatible!

For now you care about \( \int_{a}^{b} f(x) \, dx \) in \( d=2 \), i.e. smooth function on periodic domain (closed curve).

we will get good results using equal weights \( w_j \) and equally-spaced (in arclength) \( x_j \).

\[ w_j = \text{arclength } \Delta s \text{ between nodes, } x_j. \]

\[ \text{Order of convergence then depends on smoothness of } f. \]

\[ \text{It can be shown if } f \text{ is analytic function,} \]

\[ \text{then convergence is exponential, i.e. exceeds any order } p! \]

\[ \text{error } = O(e^{-\alpha M}) \quad \alpha \to \infty \text{ related to distance from strip can be continued analytically into a strip around real axis.} \]

In HW4 you'll find it convenient to parametrize by angle \( \Theta \) not arclength is.

Then \( \int_{2\pi} f(x) \, ds = \int_{0}^{2\pi} f(\Theta) \, ds \)

\[ \text{arclength change variable. } \quad w_j = ds/d\Theta \cdot \Delta \Theta \]

thus if \( ds/d\Theta \) is as smooth as \( f \), you retain same convergence. \( w_j \) weights.

Next time we'll apply this to solve BIE: Nyström method.
Numerical integration: more about quadrature

**TRAPEZOID RULE**

Given \( g(t) \) function, want integral over closed interval \([a,b]\) equally-spaced points labeled \( j = 0, \ldots, N \) spaced \( h = \frac{b-a}{N} \), \( t_j = a + jh \)

\[
\int_a^b g(t) \, dt \approx h \left[ \frac{1}{2} g(t_0) + g(t_1) + g(t_2) + \ldots + g(t_{N-1}) + \frac{1}{2} g(t_N) \right]
\]

This is a just sum of areas of trapezoids

What is order of convergence?

Define error (remainder)

\[
R[g] := \int_a^b g(t) \, dt - h \left[ \frac{1}{2} g(t_0) + g(t_1) + \ldots + g(t_{N-1}) + \frac{1}{2} g(t_N) \right]
\]

Intuitively, if \( g \) is smooth, then area error for each trapezoid is like segment of circle, radius like \( g \) -

\[
\text{area} \approx \frac{h^2}{2} \approx h^2 \frac{1}{2} g''(c) \Rightarrow \text{total error} \approx Nh^2 \frac{1}{2} g''(c) \approx 0 \left( \frac{1}{h^2} \right) g'' = O(h^2) g''
\]

**Thm:** Let \( g \in C^2[a,b] \), then \( |R[g]| \leq \frac{1}{12} h^2 (b-a) \left( \| g'' \|_{L^\infty} \right) \)

**Proof:** consider region \([t_0, t_1]\), define 'Peano' kernel here \( k(t) := \frac{(t-t)(t_1-t)}{2} \)

Then \[
\int_{t_0}^{t_1} k(t) g''(t) \, dt = - \int_{t_0}^{t_1} k' g'(t) \, dt = - \left[ k g \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} k'' g \, dt
\]

\[
k' = t_1 \epsilon_{t_0} - 2t \quad k'' = -1
\]

\[
= h \left[ g(t_0) + g(t_1) \right] - \int_{t_0}^{t_1} g(t) \, dt
\]

Summing over all intervals \([t_i, t_{i+1}]\) gives, with \( k(t) := \frac{1}{2} (t-t_i)(t_{i+1}-t) \quad \forall \quad t_i < t < t_{i+1} \)

\[
\int_a^b k(t) g''(t) \, dt = - R[g]
\]
since \( k(t) \) nonnegative on \([a, b]\)

\[
\left| R[g] \right| = \left| \int_a^b k g'' \, dt \right| \leq \| k \|_{L^1} \| g'' \|_{L^\infty}
\]

\[
\Rightarrow \int_a^b k(t) \, dt = N \int_0^{b/2} t(1-t) \, dt = N \left( \frac{b^3}{2} - \frac{b^4}{3} \right) = b^2(b-a)/12
\]

Remarks: this quadrature rule is in the form \( \sum_{j=1}^N w_j g(t_j) \)

- In some sense the order \( O(N^{-2}) \) is due to treatment of the ends of interval.
  It is possible to get higher-order with more complicated weight near ends,
  e.g. Simpson, Gaussian quad, ... beautiful.

- We really care about periodic intervals, where there are no end effects.
  As mentioned, for smooth (analytic) functions, a simple equal-spaced,
  equal-weight scheme gives exponential \( O(e^{-KM}) \) convergence!

Let's postpone proofs to another section.

**Nyström Method**

\[
(K - I) \tau = f
\]

Fredholm 2nd kind

\[
\sum_{j=1}^N w_j K(t, t_j) \tau(t_j) \quad \text{quadr.} \quad \approx \quad \sum_{j=1}^N w_j K(t, t_j) \tau(t_j)
\]

Must hold at each \( s = t_j \):

\[\forall t_j \in \left[ a, b \right]: \sum_{j=1}^N w_j K(t, t_j) \tau(t_j) - \tau(t_j) = f(t_j) \quad \text{for}\quad i = 1, \ldots, N, \]

\[\text{N} \times \text{N matrix } \begin{bmatrix} (K)_{ij} \end{bmatrix} \quad \text{of solution vector } \tau \in \mathbb{R}^N \quad \text{of vector } f \in \mathbb{R}^N\]

\[= \left[ (K)_{ij} - \delta_{ij} \right] \tau(t_j) = f(t_j) \quad \text{linear algebra problem} \quad \text{takes } O(N^3) \quad \text{CPU effort.}\]

Nyström's key observation was that the best way to find \( \tau(t) \) in between the \( t_j \) was:

\[\tau(t) = \sum_{j=1}^N w_j K(t, t_j) \tau(t_j) - f(t) \quad \text{i.e. to use the kernel itself to interpolate.}\]
If we had done this method to a 1st-kind I.E., would have got,
\[ \hat{K} \t = \hat{F} \]
(note the final interpretation step not possible here.)

E.g. integral kernel \[ k(s,t) = e^{-\frac{(t-s-t)}{2}} \] on \( [0,1] \)
\[ \leq C [0,1]^2 \]
\[ \sqrt{\varepsilon} \text{ width } \approx \varepsilon \]

use \( w_j = \frac{1}{N} \cdot w_j \)
\[ t_j = \frac{t + s}{2} \]

Given for \( K \) matrix exactly \( A \) matrix, \( a_{ij} = \frac{1}{N} e^{-\frac{(t-s-t)}{2}} \)
from HW1.1

Since \( K \) continuous, op \( \hat{K} : C[0,1] \rightarrow C[0,1] \) is compact.

How manifest itself numerically?

Eigenvalues converge down towards zero (exponentially fast) as \( N \to \infty \).

Spectrum in \( \Lambda \):
\[ \lambda_1 \quad \lambda_2 \]

Now you see why
2nd-kind are better. \( (A-I) \hat{x} = \hat{F} \) is well-conditioned.

reflected itself in
ill-conditioned
ERROR ANALYSIS of INTEGRATION OF PERIODIC FUNCS.

Why is crude equal-weight equally-spaced quadrature \( \int_0^{2\pi} g(x) \, dx \approx \frac{2\pi}{N} \sum_{j=1}^{N} g(\frac{2\pi j}{N}) \) so good?

**ANalytic CASE** (§1.4, Kneser "Numerical Analysis").

Then, let \( g : \mathbb{R} \to \mathbb{R} \) be analytic & \( 2\pi \)-periodic. Then there exists a strip \( D = \mathbb{R} \times (-a,a) \subset C \) with \( a > 0 \) s.t. \( g \) can be extended to a holomorphic and \( 2\pi \)-periodic bounded function \( g : D \to C \).

The error for above quadrature rule is bounded by

\[
|\text{residual}| \leq \frac{4\pi M}{e^{Na/2}}
\]

when \( M \) is a bound for holomorphic function \( g \) on \( D \).

Remark: this proves exponential convergence of errors \( O(e^{-an}) \)

**Proof:**

1st PART

Analytic \( g \) at each \( x \in \mathbb{R} \), Taylor expansion converges in some open disk, radius \( r(x) > 0 \).

This provides a \( 2\pi \)-periodic holomorphic extension of \( g \) since \( x \in \mathbb{R} \) & \( x + 2\pi n \) have same Taylor expansion.

Can cover \([0,2\pi]\) with finite \# of such disks.

\( a \) can be chosen to be any width, \( < \) minimum \( r(x) \).

\( g \) is then bounded on the strip \( D \).

2nd PART

Consider \( \cot(z) \), which has residuals (pole strengths)

of \( 1 \) at \( z_j = \pi j \), \( j \in \mathbb{Z} \) (since \( \frac{d}{dz} \cot(z) = 0 \)).

Then, \( g(z) \cot(\frac{Nz}{2}) \) has residuals \( \frac{2}{N} g(\frac{2\pi j}{N}) \)

at points \( z_j = \frac{2\pi j}{N} \).
Residue Thm gives, for \( \alpha < a, \)
\[
\int_I g(z) \cot \left( \frac{Nz}{2} \right) \, dz = 2\pi i \sum \text{residues} = \frac{2\pi i}{N} \sum_{j=1}^{N} g \left( \frac{2\pi j}{N} \right)
\]  
(1)

Schwarz reflection principle: \( g \) red on \( R \) so \( g(\bar{z}) = g(z) \)
the imaginary part is antisymmetric in \( \Im z \).

\( \Rightarrow \) this integral becomes
\[
\int_{i\alpha + 2\pi}^{i\alpha} 2 \text{Im} \ g(z) \cot \left( \frac{Nz}{2} \right) \, dz = 2i \Re \int_{i\alpha}^{i\alpha + 2\pi} i g(z) \cot \left( \frac{Nz}{2} \right) \, dz
\]

Using (1),
\[
\Re \int_{i\alpha}^{i\alpha + 2\pi} i \cot \left( \frac{Nz}{2} \right) g(z) \, dz = \frac{2\pi i}{N} \sum_{j=1}^{N} g \left( \frac{2\pi j}{N} \right)
\]
our quadrature rule!

Cauchy integral then,
\[
\oint_C g(z) \, dz = 0 \quad \text{since analytic in } \mathcal{D}
\]
so
\[
\Re \int_{i\alpha}^{i\alpha + 2\pi} g(z) \, dz = \int_{0}^{2\pi} g(r) \, dr
\]
\( \Rightarrow \) error
\[
\left| R_n[g] \right| = \left| \Re \int_{i\alpha}^{i\alpha + 2\pi} \left[ 1 - i \cot \left( \frac{N\alpha}{2} \right) \right] g(z) \, dz \right|
\]
\[
\left| 1 - i \cot \frac{N\alpha}{2} \right| = \left| 1 + \frac{e^{\frac{N\alpha i}{2}} + e^{-\frac{N\alpha i}{2}}}{e^{\frac{N\alpha i}{2}} - e^{-\frac{N\alpha i}{2}}} \right| \leq \frac{2}{e^{N\alpha} - 1}
\]
Take limit \( \alpha \to a \). \( \Box \)

Remark: \( \frac{1}{\pi N} \Im \cot \frac{N\alpha}{2} \) is just an approximation to double layer potential placed along the \( \Re \) axis. \( \alpha \to 1 \). \( \Box \)

There also exist Euler-Maclaurin theorems for \( C^{2m+1} \) functions:

Thm: Let \( g \in C^{2m+1} \) be \( 2\pi \)-periodic, for some \( m \geq 1 \).

Then
\[
\left| R_n[g] \right| \leq \frac{C}{N^{2m+1}} \int_0^{2\pi} |g^{(2m+1)}(\theta)| \, d\theta \quad \text{where} \quad C = \frac{2\pi}{k^{2m+1}}
\]

Proof requires Bernoulli poly's (see Kress 59-61).

Smother \( g \) \( \Rightarrow \) higher-order convergence.
Scattering theory

**Exterior Helmholtz problem**

\[
\begin{aligned}
& (\Delta + k^2) u^s = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{A} \\
& u^s = f \quad \text{on} \quad \partial A \\
& \frac{\partial u^s}{\partial r} - iku^s = o\left(\frac{1}{r}\right) \quad \text{Sommerfeld radiation condition}
\end{aligned}
\]

Says: only outward-going waves persist at large distances.

We will show, given \( f \), the above has unique solution \( u^s \).

**Scattering:** if \( u^i \) is incident field, \( e^{ikd \cdot x} \) plane wave

choose \( f = -u^i \) on \( \partial A \)

then total field \( u = u^i + u^s \)

obey \( (\Delta + k^2) u = 0 \) in \( \mathbb{R}^d \setminus \overline{A} \)

\( u = 0 \) on \( \partial A \) — Dirichlet reflecting BCs.

**Fundamental solutions**

\[
\Phi(x,y) = \begin{cases} \\
\frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) & d=2 \\
\frac{e^{ik|\mathbf{x} - \mathbf{y}|}}{4\pi |\mathbf{x} - \mathbf{y}|} & d=3
\end{cases}
\]

\( H_0^{(1)} \) Hankel func. \( = I_0 + iY_0 \)
Scattering Theory

Wave equation \( U(x,t) \) time-dependent, e.g. acoustic pressure field, (real-valued)

\[
U_{tt} = c^2 \Delta U \quad \text{in } \mathbb{R}^d \quad (\text{WE})
\]

Time-harmonic \( U(x,t) = \text{Re} \left[ e^{-i\omega t} U(x) \right] \)

\( \omega = (\text{angular}) \text{ frequency} \) rotating exponential \( \rightarrow \) complex, stationary.

Subst. (1) in (WE): \( \text{Re} (-\text{Im}) e^{-i\omega t} U \Rightarrow c^2 \text{Re} e^{-i\omega t} \Delta u \quad \forall t \quad \Rightarrow (\Delta + k^2) u = 0 \) - Helmholtz Eqn.

Flux: flow of energy

(WE) says a beautiful conservation law: \( \frac{\partial}{\partial t} \int_{\Omega} E(x,t) \, dx = -\int_{\partial \Omega} \vec{N} \cdot \vec{F}(y,t) \, dy \)

"rate of change of energy = flux leaving region."

How?

Multi. WE by \( U_t \): \( \frac{\partial}{\partial t} \int_{\Omega} E(x,t) \, dx = -\int_{\partial \Omega} \vec{N} \cdot \vec{F}(y,t) \, dy \)

\[
\frac{1}{2} (U_t^2)_t = \nabla \cdot (U_t \nabla U) - \frac{\nabla U_t \cdot \nabla U}{\frac{1}{2} |\nabla U|^2}
\]

so \( \frac{\partial}{\partial t} \frac{1}{2} (U_t^2 + k^2 |\nabla U|^2) = -\nabla \cdot \left( -c^2 U_t \nabla U \right) \)

defines \( E(x,t) \)

defines \( F(x,t) \)

Integrate over any \( \Omega \) and apply Divergence Thm. proves the conservation law (integral form).

What is flux for static field \( u(x) \)?

Energy can oscillate in \( k \) out of region \( \Rightarrow \int F(x) \, dx = \frac{\omega}{2\pi} \int_0^{2\pi} F(x(t)) \, dt =: \left< \vec{F}(x) \right> \) net flux integral over one period (on time-average).
\[ F(x) = -c^2 \partial_t^2 u - c^2 \text{Re} \left[ -i \omega e^{-i \omega t} \right] \text{Re} \left[ e^{i \omega t} \overline{u} \right] \]
\[ = -\frac{\omega^2}{4} \left[ -i e^{-2i \omega t} \overline{u} \partial_t u + i u \overline{\partial_t u} - i u \overline{u} + i e^{2i \omega t} \overline{u} \partial_t u \right] \]
\[ \text{vanish on time average} \]
\[ \text{so} \quad f(x) = -\frac{\omega^2}{2} \text{Im}[u \overline{\partial_t u}] \]

Note \( \langle \mathcal{E}(x,t) \rangle = \frac{1}{2} |u|^2 + \frac{\omega^2}{2} |\overline{\partial_t u}|^2 \) \( \text{mean} \overline{u} \cdot \overline{\partial_t u} \)

- this is (square of) H₂ Sobolev "energy" norm.

**Radiation condition:**

Solution \( u^s \) to Helmholtz Eqn. in some region including exterior of some large sphere

is "radiating" if

\[ \lim_{r \to \infty} r^{d+1} \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0 \]

holds uniformly in all directions \( \hat{r} \),

\[ r := |x| \]

Sommerfeld condition (1912)

Ensures all flux is outward.

\[ r \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = r \left( \frac{e^{ikr}}{r} + i ke^{ikr} - i ke^{ikr} \right) \to 0 \text{ as } r \to \infty. \]

So \( e^{ikr} \) is radiating but \( e^{-ikr} \) is not!

This condition ensures (proof = Sommerfeld, see Kress) uniqueness for:

Exterior Helmholtz BVP

\[ \left\{ \begin{array}{l}
(D + k^2) u^s = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \Omega \\
\lim_{r \to \infty} r^{d+1} \left( \frac{\partial u^s}{\partial r} - k u^s \right) = 0 \quad \text{uniformly in angle}.
\end{array} \right. \]

"What is field due to radiating body?"

**Scattering problem:**

\[ \left\{ \begin{array}{l}
(D + k^2) u = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\Omega}
\\
u = 0 \quad \text{on} \quad \partial \Omega
\\u^s \text{ is radiating.}
\end{array} \right. \]

\( u^s \) solves Helmholtz itself

Scatt prob. solved by finding \( u^s \) solution to ext. Helmholtz with \( f = -u^i \) on \( \partial \Omega \).
Fundamental Solution

\[ \Phi(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & d=2 \\ \frac{e^{ik|x-y|}}{4\pi|x-y|} & d=3 \end{cases} \]

Remarks

1) \( H_0^{(1)}(z) = J_0(z) + iY_0(z) \sim \frac{1}{\sqrt{\pi z}} e^{i(2-\frac{1}{2})} \{ 1 + O(\frac{1}{z}) \} \) as \( z \to 0 \)

Hankel & Bessel & Neumann wafer, decaying ampl. like \( r^{-\frac{1}{2}} \)

For \( r \to 0 \) in \( d=2 \) note \( \Phi(x,y) = \frac{i}{2\pi} \ln|x-y| + O(1) \)

Same singularity as fund. soln. for Laplace eqn. (clearly true in \( d=3 \) too).

Since singularity same, can show all Jump Relations are same as before.

ii) Your computer knows \( H_0^{(1)}(z) \) (math libraries, Matlab, etc.).

Green's Repuls Formulas: Let \( u \) be a Helmholtz solution, then

\[ u(x) = \pm \int_{\partial \Omega} \left[ u(y) \Phi(x,y) - u(x) \frac{\partial \Phi(x,y)}{\partial n_y} \right] ds_y \quad \text{for } x \in \Omega \setminus \text{outside } \Omega \]

\( \cdot \) Interior GRF: \( x \in \Omega \), \( u \) means \( u^i \) i.e. from inside.

\( \cdot \) Exterior GRF: \( x \in \mathbb{R}^d \setminus \Omega \), \( u \) means \( u^e \) from outside, \( u \) must be radiating solution.

The radiation condition ensures there's no 'boundary term' at \( \infty \).

\( \text{Proof:} \) same as GRF for Laplace operator, using fact that singularity is the same.

First show \( \int_{\partial \Omega} |u|^2 \, ds = O(1) \) as \( r \to 0 \)

\[ \int_{\partial \Omega} \left( \frac{\partial u}{\partial n_r} - iku \right)^2 \, ds = \int_{\partial \Omega} \left( \frac{\partial u}{\partial n_r} - k |u|^2 + 2k \text{Im} \frac{u}{\partial n_r} \right) \, ds \]

vanishes if radiating, as \( r \to 0 \)

Thus in any region \( R \) in which \( u \) a solution, have flux balance (no net flux), since

\[ \text{Im} \int_R u \bar{u}_n \, dx = \int_R u \frac{\partial u}{\partial n} + \frac{i}{i} \frac{\partial u}{\partial n} \, dx \quad \text{by GT} \]

\[ = 0 \quad \text{purely real} \]
Apply flux balance to \( R = B \setminus \bar{\Omega} \) gives
\[
\int_{\partial \Omega} 2k \text{ Im } u \frac{\partial u}{\partial r} \, ds = \int_{\partial \Omega} 2k \text{ Im } u \bar{u}_n \, ds
\]
\[
\text{some finite number } F
\]
Combine with (1) gives
\[
\lim_{r \to 0} \int_{\partial \Omega} \left[ \frac{\partial u}{\partial n} \right]^2 + k |u|^2 \, ds = -F
\]
\[
\text{sum of nonnegative terms so each must be bounded}
\]
\[
\lim_{r \to 0} \int_{\partial \Omega} |u|^2 \, ds = O(1)
\]
Now take sphere surface term in GRF, show vanishes as \( r \to 0 \):
\[
\int_{\partial \Omega} \left[ u \frac{\partial \Phi}{\partial y} - u \Phi \right] \, ds
\]
\[
= \int_{\partial \Omega} u \left[ \frac{\partial \Phi}{\partial n} - i k \Phi \right] \, ds
\]
\[
= \int_{\partial \Omega} \Phi \left[ u - i k \bar{u} \right] \, ds
\]
\[
= : I_1
\]
\[
\text{for } x \in B \setminus \bar{\Omega}
\]
\[
\Phi(x, \cdot ) \text{ radiating}
\]
\[
I_1 \leq \sqrt{\int_{\partial \Omega} |u|^2 \, ds} \sqrt{\int \left[ \frac{\partial \Phi}{\partial n} - i k \Phi \right]^2 \, ds}
\]
\[
\to 0 \text{ as } r \to 0
\]
\[
O(1)
\]
\[
O(1)
\]
since surface area is \( \mathcal{C}_1 \, d-1 \)
\[
\Phi(x, \cdot ) = O\left( \frac{1}{r^{d-1}} \right)
\] and \( u \) radiating \( \Rightarrow I_1 \to 0 \) as \( r \to 0 \) \( \text{(integrated, uniformly-1)} \)

Finally, applying Interior GRF to \( B \setminus \bar{\Omega} \) gives
\[
u(x) = -\int_{\partial \Omega} u_n(\Phi(\Omega)) \, ds
\]
\[
\text{bounded domain } x \in B \setminus \bar{\Omega}
\]
\[
\text{just shown vanishes as } r \to 0
\]
\[
\text{This was proved by} \text{ Wilcox (1956)} \ldots \text{see Colton \& Kress "Inverse..." book Thm. 2.4.}
\]

**Boundary Integral Eqns:**

The crude way to solve exterior Helmholtz BVP is pure double-layer representation:
\[
x \in \mathbb{R}^d \setminus \bar{\Omega}, \quad \bar{u} (x) = (DT) (x)
\]
\[
\text{we want } u^* = f = -u^i_{\text{inc}} \text{ incident field.}
\]

Thus for a scattering problem if \((\mathbb{D} + \frac{i}{2}) \varphi = -u^i_{\text{inc}}\)

Typically \( u^i_{\text{inc}} = e^{i k \mathbb{N} \cdot x} \), a plane wave.

Next time: why does \( \mathbb{D} + \frac{i}{2} \) go singular at some \( k \)?
total field \( u = u^i + u^s \)

\( k = \text{wave number} \)

Radiation solved by finding \( u^s \), solving interior Dirichlet BVP for Helmholtz eqn:

\[
\begin{align*}
(\Delta + k^2) u^s &= 0 \quad \text{in } \mathbb{R}^d \backslash \overline{\Omega} \\
\lim_{r \to \infty} r^{-1/2} (\frac{\partial u}{\partial r} - i k u^s) &= 0, \quad \text{on } \partial \Omega
\end{align*}
\]

One way to measure \( u^s \) is by its 'far-field pattern' \( u^s(\hat{r}) \):

\( \hat{r} \to \infty \)

Then: Every radiating soln. to the Helmholtz eqn. has asymptotic behavior of outgoing spherical wave

\[
u^E(x) = \frac{e^{ik|x|}}{|x|^d} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}
\]

as \( |x| \to \infty \).

Proof (d=3 case)

\[\Phi(x,y) = \frac{e^{ik|x-y|}}{|x-y|}\]

\( k = \text{wave number} \)

\[\left|x-y\right| = \sqrt{x^2 - 2xy + y^2} = |x| - |y| + O\left(\frac{1}{|x|}\right)\]

\[
e\frac{e^{ik|x-y|}}{|x|} = e^{i k|x|} \left\{ e^{-i k|x-y|} + O\left(\frac{1}{|x|}\right) \right\}
\]

\[
\frac{\partial}{\partial n_y} e^{ik|x-y|} = \frac{e^{ik|x|}}{|x|} \left\{ \frac{\partial}{\partial n_y} e^{-ik|x-y|} + O\left(\frac{1}{|x|}\right) \right\}
\]

Insert these into GFR, proved last time for radiating solutions:

\[
u(x) = \frac{e^{i k|x|}}{|x|} \left\{ \frac{1}{4\pi} \int_{\partial \Omega} \left[ u(y) \frac{\partial}{\partial n_y} e^{-ik|x-y|} - u_\infty(y) e^{-ik|x-y|} \right] dy + O\left(\frac{1}{|x|}\right) \right\}
\]

Identify as \( u_\infty(x) \) in Thm.

For \( d = 2 \) we use \( \Phi(x,y) = \frac{1}{2\pi} H_0(0)(ikx-y) \quad \text{with} \quad H_0(0) = \sqrt{\frac{2}{\pi k}} e^{i(\pi - \frac{1}{2})} \left(1 + O\left(\frac{1}{|x|}\right)\right) \)

Similar proof to above given, identify as \( u_\infty(x) \)

\[
u(x) = \frac{e^{i k|x|}}{|x|^{1/2}} \left\{ \frac{e^{i\pi/4}}{2\pi k} \int_{\partial \Omega} \left[ u(y) \frac{\partial}{\partial n_y} e^{-ik|x-y|} - u_\infty(y) e^{-ik|x-y|} \right] dy + O\left(\frac{1}{|x|}\right) \right\}
\]
Interpretation: $U_0(x)$ obtained by (weighted) integrals of $u \cdot \nu$ on $\partial \Omega$.

Outgoing flux to $\infty$ is

\[ \sim \int_{\partial \Omega} \text{Im}(\nu \cdot \nabla u_0) \, d\sigma \sim \int_{\partial \Omega} \text{Im} \left( \frac{e^{ik(x-y)}}{\sqrt{4\pi k}} \right) \, d\sigma \]

use far field rep.

\[ \frac{1}{R^2} \nu \cdot \frac{k \nu}{R^2} u_0 \]

\[ \sim \int_{\partial \Omega} |U_0(x)|^2 \, d\sigma \]

integral of power radiated over all angles.

Given double-layer rep. for $u^s$, how do you find $u_0$?

$u^s(x) = \mathcal{D}(\tau)(x)$ for $x \in \mathbb{R}^2 \setminus \overline{\Omega}$

recall $\tau$ found by $\nabla E_\tau$, $(\nabla + \frac{1}{i} \nu) \tau = -u^i$ on $\partial \Omega$.

As above, consider $|y| \to \infty$:

$u^s(x) = \int_{\partial \Omega} \frac{\hat{e}_x \cdot \nabla \tau(y)}{\sqrt{4\pi k}} \tau(y) \, ds_y = \frac{e^{ik|x|}}{\sqrt{4\pi k}} \int_{\partial \Omega} \frac{\hat{e}_x \cdot \nabla \tau(y)}{\sqrt{4\pi k}} \tau(y) \, ds_y$

$\Rightarrow$ $u_0(x) = \frac{e^{-ik|x|}}{\sqrt{8\pi k}} \cdot ik \int_{\partial \Omega} \tau(y) \frac{\hat{e}_x \cdot \nabla \tau(y)}{\sqrt{4\pi k}} \, ds_y$

This is (3.6) in Korn's Kill review

for $\gamma = 0$ case

In practice, once you have $\mathcal{N}$ at the boundary points, multiply each by the geometric factor $(\hat{\nu}_y \cdot \hat{x}) e^{-i k |x|}$ and use same quadrature as usual.
Consider Titchmarsh eigenvalue problem
\[ \begin{align*}
-\Delta u &= k^2 u \quad \text{in } \Omega \\
u_n &= 0 \quad \text{on } \partial \Omega
\end{align*} \]

non-trivial \( u_j = \text{eigenfunctions} \)
\[ k_j = \text{Eigenvalues (or just eigenvalues)} \quad j = 1, 2, \ldots \infty \]

\( u_j \) solve Helmholtz eqn. \( (\Delta + k_j^2) u_j = 0 \) \( \text{in } \Omega \)

So they could be represented by single-layer potential \( u(x) = (S\delta)(x), x \in \Omega \)

JKL then says, \( u_0 = D^T \vec{e} + \frac{1}{2} \vec{e} \quad \text{for limiting value just inside boundary} \)

Neumann BCs mean LHS is zero \( \Rightarrow (D^T + \frac{1}{2}) \vec{e} = 0 \) for some nonzero \( \vec{e} \)

The operator \( D^T + \frac{1}{2} \) has nontrivial nullspace when \( k = k_j \)

Fredholm theory gives us \( \dim \text{Null} (I - D) = \dim \text{Null} (I - D^T) \quad \text{for } D \text{ compact} \)

So \( D + \frac{1}{2} \) is not invertible when \( k = k_j \)

\( \Rightarrow \) Our double-layer BIE for scattering, \( (D + \frac{1}{2}) \tau = -u^i \), fails at \( k = k_j \).

The Fix:

'Mix an imaginary amount of single-layer \( \eta \)' \( \Re \tau \)

\( \tau = (D - i\eta S) \tau \quad \text{choose } \eta > 0 \)

optimal, Kress suggests \( \eta = k \)

BIE becomes \( (D - i\eta S + \frac{1}{2}) \tau = -u^i \)

\( \tau \) is not singular for any real \( k \rightarrow 0 \) (next time)

Recall \( S(x,y) \) has log singularity, so to get accurate Nyström method, need special quadrature.
QUADRATURE RULES for LOG SINGULARITY ... in short

Interlude on Interpolation:

given cont. Func. \( f \); approximate by

\[
\sum_{k=0}^{2n-1} a_k \phi_k = : f_n \quad \in \text{Span} \{ \phi_k \}
\]

\( \text{We want } f_n \text{ to match } f \text{ at } 2n \text{ collocation points } \{ t_j \}_{j=0}^{2n-1} \)

so

\[
\sum_{k=0}^{2n-1} a_k \phi_k(t_j) = y_j \quad j = 0 \ldots 2n-1
\]

If matrix \( A = (a_{ij}) \) nonsingular then \( \{a_{ij}\} \) unique for any set \( \{y_j\} \)

i.e. interpolation.

There are many possible sets of \( \phi_k \), e.g.

- polynomials \( t^k \)
- piecewise polynomials (splines)
- trigonometric polynomials \( \{ \cos kt \} \text{ or } \{ \sin kt \} \)

set of \( t_j \), e.g.

uniformly spaced, 'graded mesh', etc.

E.g. trig. polynomial on \([0, 2\pi]\) periodic func:

\[
f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) + \frac{a_n}{2} \cos nt
\]

Choose \( t_j = \frac{j\pi}{n} \), \( j = 0 \ldots 2n-1 \), uniformly spaced.

Analytic formulae for coeffs given \( y_j \) function samples:

\[
\begin{align*}
a_k &= \frac{1}{n} \sum_{j=0}^{2n-1} y_j \cos jk t_n \\
b_k &= \frac{1}{n} \sum_{j=0}^{2n-1} y_j \sin jk t_n
\end{align*}
\]

Why? Fourier inversion

start with

\[
\sum_{k=0}^{2n-1} e^{i j k \frac{2\pi}{n}} = \left\{ \begin{array}{ll} 1 & \text{for } j = 0 \\ 2n & \text{Kronecker delta} \\ \frac{1 - (e^{i \frac{2\pi}{n}})^j}{1 - e^{i \frac{2\pi}{n}}} & j \neq 0 \end{array} \right.
\]

e.g. sum, numerator vanishes.
Principles of successful coding (Alex's tips):

- Sit down away from computer & decide in what order things get done. Draw flowchart, etc: setup loop ← fill matrix ← solve Ax=b ← plot answer.

- Write modular code. Modules are blocks of code which talk to each other minimally & perform a defined task.
  - eg. functions/subroutines ... useful since can call repeatedly (in a loop).
  - before you code, think about the interface. E.g. the way we set up dipole.m in HW1 had well-considered inputs & outputs.

- Make code (modules) reflect the mathematics. E.g. dipole.m corresponded to one equation from the theory, but knew nothing about N, the shape, etc.

- Put all user parameters at top of code, & make everything depend on them. E.g. N=50; should be set once, prior to computing
  - E.g. \( f(x) = 1 + 0.3 \cos 3x \) should be defined once.
  - Or call \( x \) for generality.

- Test each step as you go; be creative in devising a test with a known answer. Observing that there's no crash is not a test! E.g. set \( x = 0 \), gives a circle, which you can solve analytically.

- Think about making an easy-to-use package for the user (you, your future self, or others).

- Look at other code examples (websites, tutorials, books, classmates, peers).
    - "Intro to PDE with MATLAB", J. Cooper book.

- Plot everything to check it is beautiful; plots attract attention!
More on interior resonance fix:

Recall BIE \((I + \frac{1}{k}) \tau = f\) fails as \(k \to k_j\), since \((I + \frac{1}{k}) \sigma = 0\) defines interior Neumann eigenvalues \(\lambda_j\).

Let's watch this happen:

- eigenvalues \(\lambda_j\) of \(I\) in \(C_+\) emerge from origin, and
- \(\frac{\pi}{2} - \frac{1}{2}\) as \(\omega = -\frac{1}{k}\) is increased.

- After hitting they sail around and condense on circle radius \(k_j\).

This means \(2D\) is approximately unitary in some subspace of dimension \(\approx N(k) = \#\{j: k_j < k\}\).

Why? Project idea.

\((\text{semiclassical}).\) 

we will (can) as \(k \to 0\) this scales like volume(D) \(\sim k^d\)

Note each \(\lambda_j\) passes through \(-\frac{\pi}{2}\)!

Why? Interior Dirichlet eigenmodes \(\{(a + k_j^2)u_j = 0 \quad \text{in} \quad \Omega\}

\quad \quad \text{on \(\partial \Omega\))

\(\text{JRA: Limiting value on \(\partial \Omega\), approaching from inside is}\)

\(u^- = (I - \frac{1}{k_j}) \tau\)

if \(\partial \Omega\) rep'd by double-layer potential. So eigenmodes have \((I - \frac{1}{k_j}) \tau = 0\)

Therefore a \(\tau \to -\frac{1}{2}\) when \(k \to k_j\)

This is a popular way to find eigenmodes. Try it! (HW3).

The fix: use representation \(cu(x) = ((I - \frac{1}{k_j} S) \tau)(x)\)

\text{outside} \(\Sigma\). \(\gamma = \text{some constant} > 0\).

\(\text{JRA gives BIE} \quad \quad (I - \frac{1}{k_j} S + \frac{1}{k_j}) \tau = f\)

Brakhage-Werner, Leis, Panich 
(1969's).

\(\text{Why never singular?} \quad \text{(see Cotter-Kress "Inverse..." book, p.48-49, 2"(Ed).)}\)

Suppose \((I - \frac{1}{k_j} S + \frac{1}{k_j}) \tau = 0\)

we wish to show \(\tau \equiv 0\) follows, i.e \((I - \frac{1}{k_j} S + \frac{1}{k_j}) \text{ is injective.}\)

so \(u = (I - \frac{1}{k_j} S) \tau\) has \(u^+ = 0\) by construction of BIE.

\(\Rightarrow u^+ = 0\) in all of \(R^d \setminus \Sigma\) outside \(\Sigma\), by uniqueness of exterior Dirichlet problem.

use jumps in \(u^+\) on \(\partial \Sigma\)

\(\text{JRA: \(u^- = -\tau\)}\)

\(\text{JRA, J: \(u^- = -i\eta \tau\)}\)

\(\text{why?}\)
\[ \text{GT1 applied inside } \Omega \text{ gives } \int_{\partial \Omega} \bar{u} \cdot n \, ds = \int_{\Omega} u \Delta \bar{u} + \nabla u \cdot \nabla \bar{u} \, dx \]

\[ \text{from above} \]

\[ \int_{\partial \Omega} |\bar{u}|^2 \, ds \]

\[ \int_{\Omega} -|\bar{u}|^2 + |u|^2 \, dx \]

\[ \text{prove real } \]

Take Im part of eqn shows \( \tau = 0 \). \( \Box \)

Essentially we have shown \( \int_{\partial \Omega} |\bar{u}|^2 \, ds \) is flux entering domain \( \Omega \), but this vanishes for a Helmholtz system everywhere in domain.

**Remarks**

- This is both an analytic tool to prove existence/uniqueness of scattering solutions and numerical.
- I believe sign of \( \Phi \) immaterial for numerical purposes.
- Watch fixed eigenvalues \( \Lambda \) of \( D - \Phi S \) move... they avoid \(-\frac{1}{4}\) like crazy.

---

**Modifying Nystrom method for \( D - \Phi S \):**

- \( a = 2 \)

Recall:

\[ \Phi(x,y) = \frac{1}{2\pi} \ln \left| \frac{x-y}{x-y+1} \right| \]

parameterize \( \Omega \) by \( x(t) \), \( t \in [0, 2\pi] \)

\[ dx = |x'(t)| \, dt \]

Then:

\[ (S\Phi)(s) = \int_{2\pi} \Phi(x,y) \, ds = \int_{0}^{2\pi} \frac{1}{2\pi} \ln \left| \frac{x(s) - x(t)}{x(s) - x(t) + 1} \right| \, dt \]

We want to write \( M(s,t) = M_1(s,t) \cdot \ln |s-t| \) singular + \( M_2(s,t) \)

with both \( M_1, M_2 \) analytic, and our 'in singular' function \( \Phi(s,t) \) easy to analyse.

We choose \( \ln \left( \frac{4 \sin^2 \frac{s-t}{2}}{4} \right) \) as periodic in-singular function ... it will have known Fourier coeffs.

\[ \sim 2 \ln |s-t| \text{ as } s-t \to 0 \text{, i.e. 'strength' is } 2 \]

\[ \text{note } \frac{s-t}{2} \text{ so only has } 1 \text{ singularity per period.} \]

Have:

\[ M(s,t) = \frac{-1}{2\pi} \int_{2\pi} \ln \left| x(s) - x(t) \right| \, dx(t) \cdot \ln \left( \frac{4 \sin^2 \frac{s-t}{2}}{4} \right) + M_2(s,t) \]

\[ M_2(s,t) \text{ has no singularities as } s-t \to 0 \text{, is analytic, and has } M_2(s,t) = \left[ \frac{2}{4} - \frac{1}{2\pi} \ln \left( \frac{4 \sin^2 \frac{s-t}{2}}{4} \right) \right] \, dx(t) \]
Here \( C = \lim_{p \to \infty} \left[ \frac{1}{m} \ln \left( \frac{1}{m} \ln^2 \left( \frac{1}{m} \right) \right) \right] = 0.57 \ldots \) is Euler's const.

Note you now can complete \( M_1 \) & \( M_2 \) at any \( s,t \) (use (*) for \( M_2 \)).

We may split up \( \frac{\partial \Phi}{\partial y} (x,y) \) for \( \Theta(x) \) in similar way [see Krishnamurthy].

Thus our \( BIE \) is

\[
\int_0^{2\pi} K(s,t) T(t) \, dt + \frac{1}{2} T(s) = f(s)
\]

with

\[
K(s,t) = K_1(s,t) \ln\left( 4\sin^2 \frac{s-t}{2} \right) + K_2(s,t)
\]

Quadrature:

Using uniform quadrature in \( t \) variable is what you already do (\( t = \) angle' variable so weights \( w_i = |\theta(\tau_i)/4\pi| \))

This gave exponential convergence for analytic kernels (eg., \( D(s,t) \)).

Beautiful thing: can get exponential (spectral) convergence also for above' log singularity!

Take: 2n equally spaced quadrature points \( t_j : \)

\[
\begin{array}{ccccccc}
0 & \frac{2\pi}{2n} & \frac{4\pi}{2n} & \ldots & \frac{(2n-1)\pi}{2n} & \frac{2\pi}{2n} \\
0 & 1 & \frac{2n-1}{n} & \frac{2n-1}{n} & \frac{2n-1}{n} & \frac{2n-1}{n}
\end{array}
\]

Analytic integrand

\[
\int_0^{2\pi} K_2(s,t) T(t) \, dt = \frac{1}{n} \sum_{j=0}^{2n-1} K_2(s,t_j) T(t_j)
\]

all weights constant.

Log sing. analytic

\[
\int_0^{2\pi} K_1(s,t) \ln\left( 4\sin^2 \frac{s-t}{2} \right) T(t) \, dt = 2\pi \sum_{j=0}^{2n-1} R_j^{(n)}(s) K_1(s,t_j) T(t_j)
\]

translational invariance, \( K_1^{(n)}(s) = R_0^{(n)}(s-s_j) \)

Nyström method will be, by setting \( s = t_i \):

\[
\sum_{j=0}^{2n-1} \left[ \frac{1}{n} K_2(t_i, t_j) + 2\pi R_0^{(n)}(s_i-s_j) K_1(t_i, t_j) \right] T(t_j) = T(t_i) = f(t_i)
\]

this is your new "K" matrix

Note \( K_0^{(n)}(s_i-s_j) = R_1^{(n)}(0) \) Let's now get \( R_j^{(n)}(s) \)...
Today we show:

1. Complex contour integration intimately related to Laplace double layer
2. \( \ln(z) \) is kernel of 'Neumann-to-Dirichlet' map on unit disk
3. How to interpolate periodic functions with Fourier series
4. Derive spectral (exponentially convergent) log-singularity quadrature.

**Cauchy contour integral:**

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-s} \, dz = \begin{cases} 
  \int_{\gamma} f(s) \, ds & \text{if } s \in \gamma \\
  0 & \text{if } s \notin \gamma 
\end{cases}
\]

for \( f \) analytic in \( \gamma \).

Complex contour integral converts to line integral via \( ds = e^{i\theta} \, ds \).

\[
\text{Re} \left( \frac{1}{i} \frac{1}{z-s} \right) = \frac{\cos \theta}{|z-s|} 
\]

So for real \( \theta \), real part of above Cauchy integral is

\[
= -\left( \frac{\cos \theta}{|z-s|} \right) f(z) \, ds
\]

Remarks:

1. Imaginary part of Cauchy integral is \( (\text{strongly}) \) singular, thus complex analysis allows proof of results on singular integral equations (Knopp "Linear IE's", ch. 7).
2. Cauchy theorem looks like "\( f(s) = (DF)(s) \)", i.e. tempting to think in the real case surface density is just \( \tau = f/2\pi \), and \( f = 0 \) for \( s \) outside. These are wrong since \( D \) does not include Im part. Beauty of complex case is that \( \tau \) doesn't need to be solved for.

**Unit disk: Poisson kernel**

An example of Cauchy integral, write \( \zeta = e^{i\phi} \), \( s = \rho e^{i\theta} \),

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-s} \, dz = \frac{1}{2\pi} \int_0^{2\pi} \frac{z}{z-s} f(s) \, d\theta
\]

for \( s \) inside.

The point \( \frac{1}{\sqrt{2}} \) is outside, so \( \zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{z}{z-s} f(s) \, d\theta \) also holds.
Subtract the two equations: \[ f(s) = \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{z - \bar{z}}{z - s} - \frac{\bar{z} - z}{\bar{z} - 1\bar{s}} \right] f(\xi) \, d\xi \]

\[ \text{convert to } \frac{z - \bar{z}}{z - s}, \text{ using } \bar{z} = 1 \]

\[ \frac{z(z - \bar{z}) + \bar{z}(z - s)}{(z - s)(\bar{z} - s)} = \frac{1 - |s|^2}{(z - s)^2} \]

\[ = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)} \]

---

So Poisson kernel representation of interior values of \( f \) is

\[ f(\rho, \theta) = f(s) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\theta - \phi)} f(\xi) \, d\xi \]

boundary values.

Notice this is more "user-friendly" than GRF since only boundary values (not derivatives) needed.

It solves Dirichlet interior BVP directly.

The kernel \( \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\theta - \phi)} \) is sometimes called 'harmonic measure', is \( \mathcal{G}(x, y) \) where \( \mathcal{G} \) is Green's function for the domain. It is not same as kernel of layer potentials \( D, S \), etc.

\[ \mathcal{G}(x, y) \] is the Cauchy kernel for the domain.

---

**Neumann-to-Dirichlet map:**

\[ U \text{ harmonic in } \Omega \]

\[ \Omega \]

A can be written in terms of layer potentials.

Recall 'zero flux' corollary: \( \int_{\partial \Omega} \mathbf{U} \cdot n \, ds = 0 \) for harmonic maps.

\( \Rightarrow \) domain of \( A \) is the functions \( C_0(\partial \Omega) \), zero mean on \( \partial \Omega \) (otherwise no such \( u \) exists).

Recall we may add a const to \( u \) without changing \( \mathbf{U} \), so \( \mathcal{A} u_n \) unique only up to const.

Say \( \mathbf{U} = g \), given Neumann data, represent \( u \) inside by \( u(x) = (Sg)(x) \)

**TR2:** \[ g = \mathbf{U} = (D^T + \frac{1}{2}) \mathbf{E} \]

**TR1:** \[ \mathbf{E} = (D^T + \frac{1}{2})^{-1} g \]

so \( \mathbf{E} = (D^T + \frac{1}{2})^{-1} g \)

\( \Rightarrow \) since we've restricted to \( C_0(\partial \Omega) \), inverse exists, bounded.

Combining: \( U|_{\partial \Omega} = S (D^T + \frac{1}{2})^{-1} g \)

\( \frac{\text{this is } \mathcal{A}}{\text{this is } \mathcal{A}} \)
recall \[ D(s,t) = -\frac{1}{2\pi} \cos \frac{t-s}{2} \]

see triangle \[ \frac{t-s}{2} \]
which is right by geom.

This matches \[ D(s,t) = -\frac{2\zeta(s)}{4\pi} \]
for unit disk.

So \( (D^T + \frac{1}{2})^{-1} : \mathcal{C}_0(\partial \Omega) \rightarrow \mathcal{C}_0(\partial \Omega) \) is 1-to-1, and is just 2I

So ND map is \[ A = S(D^T + \frac{1}{2})^{-1} = 2S \]
the single layer operator.

In other words \[ u(s) = \int_0^{2\pi} A(s,t)g(t)\,dt \]
with kernel \[ A(s,t) = -\frac{1}{\pi} \ln r = -\frac{1}{\pi} \ln \left(2\sin \frac{s-t}{2}\right) \]

\[ \frac{1}{2\pi} \int_0^{2\pi} \ln \left(4\sin^2 \frac{s-t}{2}\right) e^{imt}\,dt \]

\[ \left\{ \begin{array}{ll}
0 & m = 0 \\
-\frac{e^{ims}}{m!} & m \neq 0
\end{array} \right. \]

\[ e^{ims} = -\frac{1}{2\pi} \int_0^{2\pi} \ln \left(4\sin^2 \frac{s-t}{2}\right) e^{imt}\,dt \]

\[ \Rightarrow \]

\[ \frac{1}{2\pi} \int_0^{2\pi} \ln \left(4\sin^2 \frac{s-t}{2}\right) e^{imt}\,dt = \left\{ \begin{array}{ll}
0 & m = 0 \\
-\frac{e^{ims}}{m!} & m \neq 0
\end{array} \right. \]

\[ \text{Fourier coeffs (Lemma 8.21, Kress, "Lin. Int. Eq")} \]

\[ \text{Since } -m \text{ gives complex coef of } \lambda m. \]

\[ \text{Uncovering single layer source } \]

\[ \text{gives plot } \]

\[ A(s,t) \]

\[ \text{amazingly this singularity} \]
\[ \text{(which is in } L^2[0,2\pi]) \text{ has} \]
\[ \text{Fourier coeffs dying like } O(\frac{1}{m}) \]
\[ \text{same as jump discontinuity.} \]
Interpolation of functions:

Goals: given samples of far points \( t_j \), \( j = 0 \cdots 2n-1 \), reconstruct smooth approx to \( f \) everywhere.

Our approximation to \( f \), called \( f_n \), will lie in \( X_n = \text{Span} \{ \xi_k \} \), \( k = 0 \cdots 2n-1 \).

Samples are: \( y_j = f(t_j) \), \( j = 0 \cdots 2n-1 \), ie \( f_n(t) = \sum \xi_k \xi_k(t) \).

- If matrix with \( k \) entry \( \xi_k(t_j) \) is nonsingular, there is unique element \( f_n \) of \( \text{Span} \{ \xi_k \} \), which matches \( y_j \) at \( t_j \).

\[
\begin{align*}
y_j &= f_n(t_j) \\
&= \sum \xi_k(t_j) \xi_k
\end{align*}
\]

Linear map \( Lnf = f_n \) is a projection since \( L_n^2 = L_n \) since \( Lnf \) already matches \( y_j \) at points.

There is a unique element \( \xi_k \) of \( X_n \) for which \( \xi_k(t_j) = \delta_{jk} \), called 'Lagrange polynomial'.

\[\xi_k(t)\]

eg. \( X_n = \text{piecewise linear between points } \{ \xi_j \text{ (splines)} \} \).

- Note \( Lnf = \sum y_j \xi_j \) - check it matches!

Key example: 'Trigometric polynomials' on uniform grids:

\[
f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left( a_k \cos kt + b_k \sin kt \right) + \frac{a_n}{2} \cos nt
\]

\( f_n \in T_n \), \( n \)th order trig poly is actually of dim \( 2n+1 \), but \( \sin nt \) dropped since vanishes at all \( t_j \).

Fourier series representation.

Note \( k = \pm n \) terms really contribute as one; \( f_n = f_{-n} \).

\[
\begin{array}{c}
\text{Real part} \\
\text{Imaginary part}
\end{array}
\]

\( f_n = \overline{f_n} \) since \( f_n \) real.

\( \hat{f} \) in \( \mathbb{R}^{2n} \)

Ref: \( f \) in \( \mathbb{R}^{2n} \) also \( \mathbb{R} \) in real params.
\[
\begin{align*}
\text{BACKWARD MAP} & \\
& \begin{cases} 
\hat{f}_o = 2 f_0 \\
\hat{f}_k = \frac{1}{2} (a_k - i b_k) \quad \text{for } k = 0, \ldots, n \\
\hat{f}_k = \frac{1}{2} (a_k + i b_k) \quad \text{where } b_n = b_0 = 0 
\end{cases} \\
\text{FORWARD MAP} \\
& \begin{cases} 
\hat{f}_k \quad \text{defining } b_n = b_0 = 0 
\end{cases}
\end{align*}
\]

Miracle: sum of \( \exp(\frac{2\pi i k}{n}) \) over all \( k \in \mathbb{Z} \) is \( \sum_{j=0}^{2n-1} e^{\frac{2\pi i j}{n}} = 2n \delta_{k,0} \) sum over grid pts

Given useful formulae
\[
\begin{align*}
\sum_{j=0}^{2n-1} e^{i k t_j} &= 2n \delta_{k,0} \\
\sum_{k} e^{i k t_j} &= 2n \delta_{j,0}
\end{align*}
\]

Finding interpolant means getting \( f\hat{y}_j \) from \( f\hat{y}_j \), such that:
\[
y_j = \sum_{k} f_k e^{i k t_j}
\]

\[
\begin{align*}
\sum_{j=0}^{2n-1} y_j e^{-i m t_j} &= \sum_{k} f_k \sum_{j=0}^{2n-1} e^{i(k-m) t_j} \\
&= 2n f_m \quad \text{for } m = -n, \ldots, n.
\end{align*}
\]

Inversion formula
\[
f_m = \frac{1}{2n} \sum_{j=0}^{2n-1} y_j e^{-i m t_j}
\]

Note matrix \( A_{jk} = \frac{1}{2n} e^{i k t_j} \) is unitary.

Lagrange poly: \( L_k \) has \( \omega \) with Fourier coeff
\[
\frac{1}{2n} \sum_{j=0}^{2n-1} \delta_{j,k} e^{-i m t_j} = \frac{e^{-i m t_k}}{2n}
\]

\[
L_k(t) = \frac{1}{2n} \sum_{m=0}^{2n-1} e^{-i m t_k} e^{i m t} = \frac{1}{2n} \sum_{m} e^{i m (t-t_k)}
\]

\[
= \frac{1}{2n} \left[ 1 + 2 \sum_{m=1}^{n-1} \cos m (t-t_k) + \cos n (t-t_k) \right] \quad \text{check in HW3!}
\]

Now armed with all Lagrange poly's you build triji interpolant
\[
f_n(t) = \sum_{k=0}^{2n-1} y_k L_k(t)
\]
Spectral Quadrature Weights:

\[ \int_0^{2\pi} f(t) \, dt \approx \sum_{k=0}^{2n-1} y_k \int_0^{2\pi} k(t) \, dt \]
using \(k(t)\) and \(\int_0^{2\pi} e^{int} \, dt = \begin{cases} 1, & m=0 \\ 0, & \text{otherwise} \end{cases} \)

\[ = \sum_{k=0}^{2n-1} w_k f(t_k) \quad \text{with weights} \quad w_k = \frac{\pi}{n} \quad \forall k. \]

Our rule is exact for \(f \in T_n\); if not then error is bounded by interpolation error \(\|f - p_n\|\), exponentially small vs \(n\) for analytic \(f(t)\).

\[ \int_0^{2\pi} f(t) \ln(4\sin^2 \frac{s+t}{2}) \, dt \approx \int_0^{2\pi} f_n(t) \ln(4\sin^2 \frac{s+t}{2}) \, dt = \sum_{k=0}^{2n-1} y_k \int_0^{2\pi} k_n(t) \ln(4\sin^2 \frac{s+t}{2}) \, dt \]

Using \(k_n(t)\),

\[ R_k^{(n)}(s) = \frac{1}{2n} \sum_m e^{-imt_n} \int_0^{2\pi} e^{imt} \ln(4\sin^2 \frac{s+t}{2}) \, dt \]

\[ = \frac{-2\pi i}{2n} \sum_{m \neq 0} \frac{1}{|m|} e^{ims} \]

\[ K_k^{(n)}(s) = -\frac{-\pi}{n} \left[ 2 \sum_{m=1}^{n-1} \frac{1}{m} \cos(m(s-t_k)) + \frac{1}{n} \cos n(s-t_k) \right] \]

As above, have derived weights which are exact for \(f \in T_n\), exponentially convergent for \(f(t)\) analytic. This is from Kress (following Mardewson & Kriömann in COS). formula comes about (eg K's review Eqn (3.11)).