Math 123 Homework Assignment #4
Due at the end of term (optional)

Part I:

1. Let \( \{ A_n, \varphi_n \} \) be a direct sequence of \( C^* \)-algebras in which all the connecting maps \( \varphi_n \) are injective. Let \( (A, \varphi^\alpha) \) be “the” direct limit.
   
   (a) Show that \( \varphi^n(a) = \varphi^m(b) \) if and only if there is a \( k \geq \max n, m \) such that \( \varphi_{n,k}(a) = \varphi_{m,k}(b) \), and
   
   (b) Show that each \( \varphi^n \) is injective.

2. Let \( A \) be a \( C^* \)-algebra. We call \( u \in A \) a \textit{partial isometry} if \( u^*u \) is a projection. Show that the following are equivalent.
   
   (a) \( u \) is a partial isometry.
   
   (b) \( u = uu^*u \).
   
   (c) \( u^* = u^*uu^* \).
   
   (d) \( uu^* \) is a projection.
   
   (e) \( u^* \) is a partial isometry.

   (Suggestion: for (b) \( \implies \) (a), use the \( C^* \)-norm identity on \( \| uu^*u - u \| = \| u(u^*u - 1) \| \).)

Part II:

3. Give the details of the argument sketched in lecture that if \( A \) is a finite-dimensional \( C^* \)-algebra, then \( A \cong \bigoplus_{i=1}^n M_{n_i} \). (Recall that you can use the argument of Corollary BG to conclude that \( A \) has a finite-dimensional faithful representation. Then we may as well assume that \( A \subset \mathcal{K}(\mathcal{H}) = B(\mathcal{H}) \). Now apply Theorem AP to the identity representation of \( A \).)

4. Let \( T = (T_{ij}) \) be an operator in \( M_n(B(\mathcal{H})) \). Show that

\[
\| T_{ij} \| \leq \| (T_{ij}) \| \leq \sum_{ij} \| T_{ij} \|.
\]
5. Let \( M_s \) be a UHF algebra such which is not a matrix algebra. (This is automatic if \( s : \mathbb{Z}^+ \to \{2, 3, \ldots\} \).) Show that \( M_s \) is not GCR.

6. Recall the following notations. Let \( \mathbf{n} = (n_1, \ldots, n_k) \in (\mathbb{Z}^+)^k \), and \( |\mathbf{n}| = n_1 + \cdots + n_k \). View elements of \( M_{|\mathbf{n}|} = M_{n_1} \oplus \cdots \oplus M_{n_k} \) as block diagonal matrices in \( M_{|\mathbf{n}|} \). Recall that \( M = (m_{ij}) \in M_{s \times k}(\mathbb{N}) \) is called admissible if \( \sum_{j=1}^{k} m_{ij}n_j \leq r_i \) for all \( i = 1, 2, \ldots, s \). We defined \( \varphi_M : M_{|\mathbf{n}|} \to M_r \) by \( \varphi_M(T_1 \oplus \cdots \oplus T_k) = (T'_1 \oplus \cdots \oplus T'_s) \) where

\[
T'_i = m_{i1} \cdot T_1 \oplus \cdots \oplus m_{ik} \cdot T_k \oplus 0_{d_i},
\]

with \( d_i = r_i - \sum_{j=1}^{k} m_{ij}n_j \), and \( m \cdot T = \bigoplus_{j=1}^{m \text{ times}} T \).

Prove that if \( \varphi : M_{|\mathbf{n}|} \to M_r \) is a \( * \)-homomorphism, then there is a unitary \( u \in M_r \) such that \( \varphi = \text{Ad} \ u \circ \varphi_M \) for some admissible matrix \( M \).

(Suggestions: (1) View \( \varphi \) as a (possibly degenerate) representation of \( M_{|\mathbf{n}|} \subseteq M_{|\mathbf{n}|} = B(\mathbb{C}^{|\mathbf{n}|}) \) into \( M_r \subseteq B(\mathbb{C}^{r_i}) \). Use Theorem AP to write \( \varphi \) as \( \sum_i \pi^i \), where each \( \pi^i \) is an irreducible subrepresentation equivalent to a subrepresentation of \( \text{id} : M_{|\mathbf{n}|} \to B(\mathbb{C}^{|\mathbf{n}|}) \). (2) Conclude that \( \varphi = \sum_{i=1}^{s} \varphi_i \) where each \( \varphi_i \) is a \( * \)-homomorphism of \( M_{|\mathbf{n}|} \) into \( M_{r_i} = B(\mathbb{C}^{r_i}) \). Then use Theorem AP again to see that \( \varphi_i \cong \bigoplus_{j=1}^{k} m_{ij} \cdot \text{id}_{M_{n_j}} \). (3) Now show that there is a \( U \in M_{r_i} \) such that \( \varphi_i(T_1 \oplus \cdots \oplus T_k) = U(m_{i1} \cdot T_1 \oplus \cdots \oplus m_{ik} \cdot T_k \oplus 0_{d_i})U^* \).)

7. Suppose that \( (A, \{\varphi^n\}) \) is the direct limit of a direct system \( \{(A_n, \varphi_n)\} \), and that \( (B, \{\psi^n\}) \) is the direct limit of another direct system \( \{(B_n, \psi_n)\} \). Suppose that there are maps \( \alpha^n : A_n \to B_n \) such that the diagrams

\[
\begin{array}{ccc}
A_n & \xrightarrow{\varphi^n} & A_{n+1} \\
\downarrow \alpha^n & & \downarrow \alpha^{n+1} \\
B_n & \xrightarrow{\psi^n} & B_{n+1}
\end{array}
\]

commute for all \( n \). Show that there is a unique homomorphism \( \alpha : A \to B \) such that

\[
\begin{array}{ccc}
A_n & \xrightarrow{\alpha^n} & B_n \\
\downarrow \varphi^n & & \downarrow \psi^n \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

commutes for all \( n \).
8. Let $J$ be an ideal in a C*-algebra $A$. We call $J$ a primitive ideal if $J = \ker \pi$ for some irreducible representation $\pi$ of $A$. On the other hand, $J$ is called prime if whenever $I_1$ and $I_2$ are ideal in $A$ such that $I_1I_2 \subseteq J$, then either $I_1 \in J$ or $I_2 \in J$. Show that every primitive ideal in a C*-algebra is prime. (Suggestion: If $I \not\subseteq J$, then $\pi|_I$ is irreducible and $[\pi(I)\xi] = \mathcal{H}$ for all $\xi \in \mathcal{H}$.)

Part III:

9. Let $A$ be the UHF algebra $M_s$ where $s = (2, 2, 2, \ldots)$. Thus $A$ is the $C^*$-direct limit arising from the maps $\varphi_M : M_{2^n} \to M_{2^n+1}$ with $M = (2)$. Let $B$ be the $C^*$-direct limit arising from the maps $\varphi_{M'} : M_{(2^n-1, 2^n-1)} \to M_{(2^n, 2^n)}$ with $M' = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)$. In terms of Bratteli diagrams

\[
\begin{array}{ccc}
2 & 1 & 1 \\
4 & 2 & 2 \\
8 & 4 & 4 \\
\vdots & \vdots & \vdots \\
\end{array}
\]

Use Elliot’s Theorem to show that $A$ and $B$ are isomorphic.

10. Let $A$ be a $C^*$-algebra without unit. Then $\tilde{A}$ is the smallest $C^*$-algebra with unit containing $A$ as an ideal. It has become apparent in the last few years, that it is convenient to work with the “largest” such algebra (in a sense to be made precise below). For motivation, suppose that $A$ sits in $B$ as an ideal. Then each $b \in B$ defines a pair of operators $L, R \in B(A)$ defined by $L(a) = ba$ and $R(a) = ab$. Note that for all $a, c \in A$,

\[
\begin{align*}
(1) & \quad L(ac) = L(a)c, \\
(2) & \quad R(ac) = aR(c), \\
(3) & \quad aL(c) = R(a)c.
\end{align*}
\]

Define a multiplier or double centralizer on $A$ to be a pair $(L, R)$ of operators on $A$ satisfying conditions (1), (2), and (3) above. Let $\mathcal{M}(A)$ denote the set of all multipliers on $A$.

(a) If $(L, R) \in \mathcal{M}(A)$, then use the closed graph theorem to show that $L$ and $R$ must be bounded, and that $\|L\| = \|R\|$. 

\footnote{If $A$ is separable, the converse holds. It has just recently been discovered that the converse can fail without the separable assumption.}

\[\text{-3-}\]
(b) Define operations and a norm on $\mathcal{M}(A)$ so that $\mathcal{M}(A)$ becomes a $C^*$-algebra which contains $A$ as an ideal. (Use the example of $A$ sitting in $B$ as an ideal above for motivation.)

(c) An ideal $A$ in $B$ is called essential if the only ideal $J$ in $B$ such that $AJ = \{0\}$ is $J = \{0\}$. Show that $A$ is an essential ideal in $\mathcal{M}(A)$. Also show that if $A$ is an essential ideal in a $B$, then there is an injective $*$-homomorphism of $B$ into $\mathcal{M}(A)$ which is the identity on $A$.

(d) Compute $\mathcal{M}(A)$ for $A = C_0(X)$ and $A = K(\mathcal{H})$.

11. Let $\text{Prim}(A)$ be the set of primitive ideals of a $C^*$-algebra $A$. If $S \subseteq \text{Prim}(A)$, then define $\text{ker}(S) = \bigcap_{P \in S} P$ (with $\text{ker}(\emptyset) = A$). Also if $I$ is an ideal in $A$, then define $\text{hull}(I) = \{ P \in \text{Prim}(A) : I \subseteq P \}$. Finally, for each $S \in \text{Prim}(A)$, set $\overline{S} = \text{hull}(\text{ker}(S))$.

(a) Show that if $R_1, R_2 \subset \text{Prim}(A)$, then $\overline{R_1 \cup R_2} = \overline{R_1} \cup \overline{R_2}$.

(b) Show that if $R_\lambda \in \text{Prim}(A)$ for all $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} \overline{R_\lambda} = \bigcap_{\lambda \in \Lambda} \overline{R_\lambda}$.

(c) Conclude that there is a unique topology on $\text{Prim}(A)$ so that $\{ \overline{S} : S \subseteq \text{Prim}(A) \}$ are the closed subsets.

This topology is called the Hull-Kernel or Jacobson topology.

12. Consider the $C^*$-algebras

(a) $A = C_0(X)$, with $X$ locally compact Hausdorff.

(b) $B = C([0,1], M_2)$, the set of continuous functions from $[0,1]$ to $M_2$ with the sup-norm and pointwise operations.

(c) $C = \{ f \in B : f(0) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \alpha \in \mathbb{C} \}$.

(d) $D = \{ f \in B : f(0) = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \alpha, \beta \in \mathbb{C} \}$.

For each of the above discuss the primitive ideal space and its topology. For example, show that $\text{Prim}(A)$ is homeomorphic to $X$. Notice that all of the above are CCR.